CHAPTER

8

Trigonometry of Acute Triangles

Trigonometry

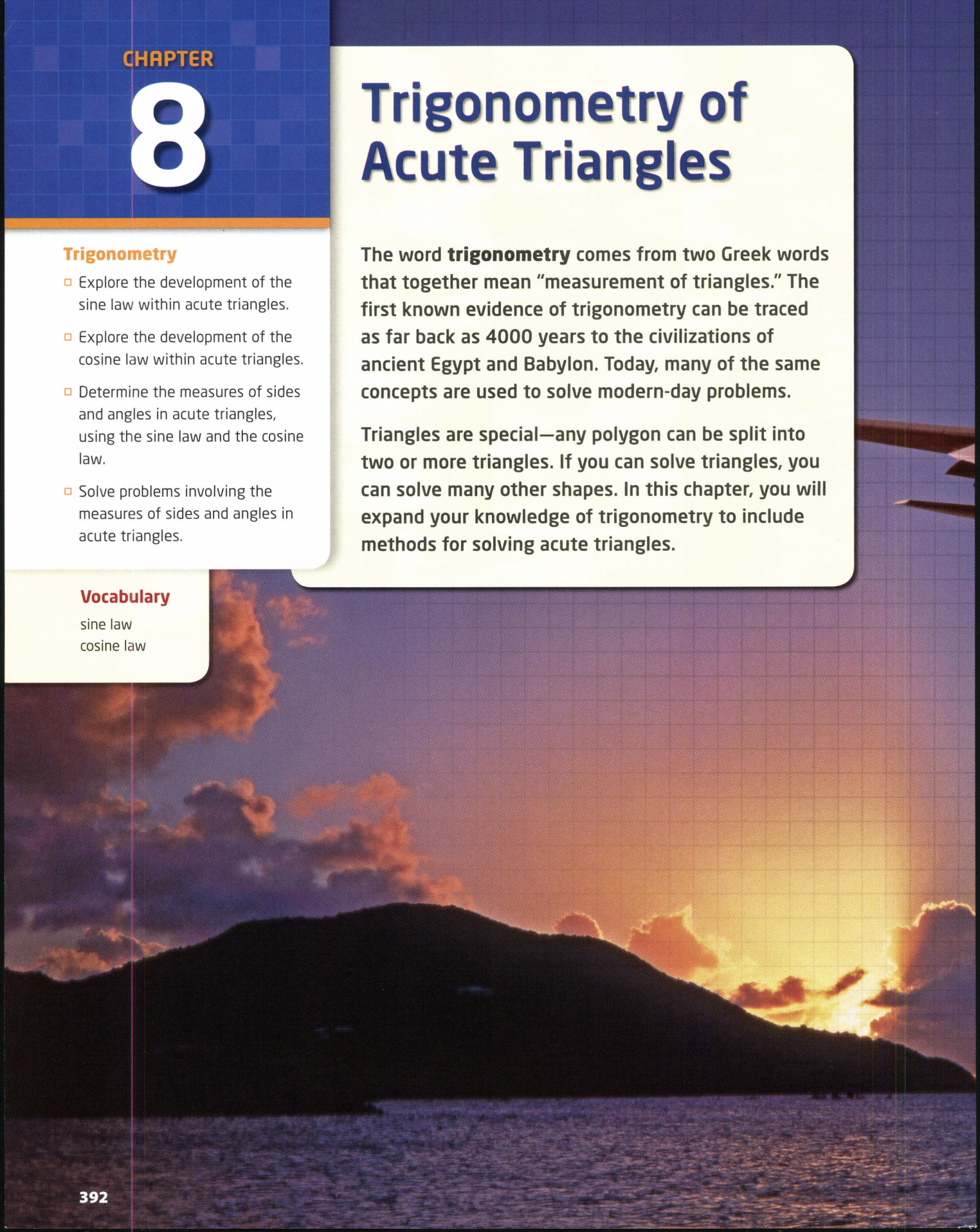
- Explore the development of the sine law within acute triangles.
- Explore the development of the cosine law within acute triangles.
- Determine the measures of sides and angles in acute triangles, using the sine law and the cosine law.
- Solve problems involving the measures of sides and angles in acute triangles.

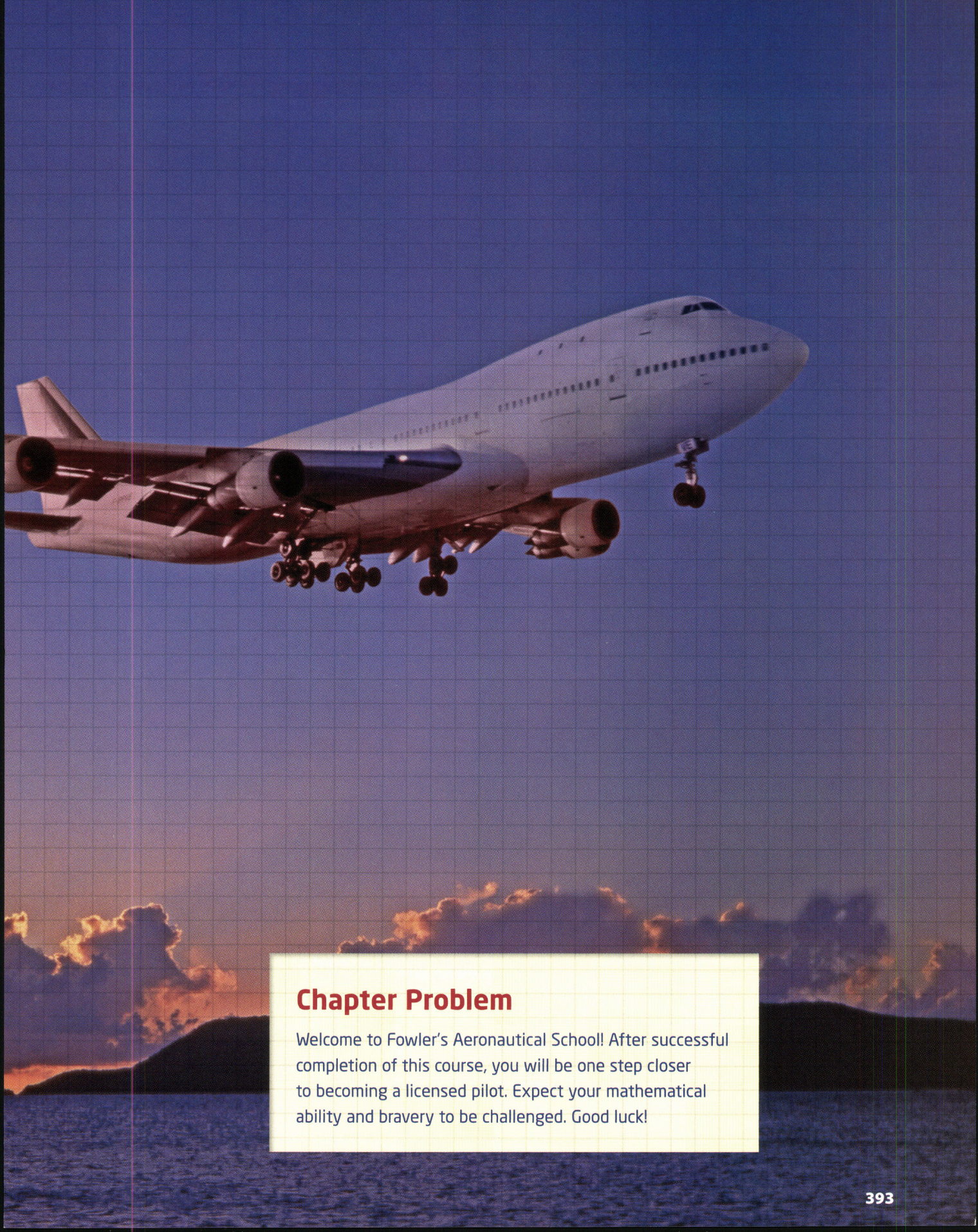
Vocabulary

sine law
cosine law

The word **trigonometry** comes from two Greek words that together mean "measurement of triangles." The first known evidence of trigonometry can be traced as far back as 4000 years to the civilizations of ancient Egypt and Babylon. Today, many of the same concepts are used to solve modern-day problems.

Triangles are special—any polygon can be split into two or more triangles. If you can solve triangles, you can solve many other shapes. In this chapter, you will expand your knowledge of trigonometry to include methods for solving acute triangles.





Chapter Problem

Welcome to Fowler's Aeronautical School! After successful completion of this course, you will be one step closer to becoming a licensed pilot. Expect your mathematical ability and bravery to be challenged. Good luck!

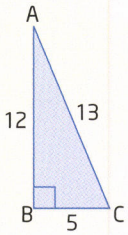
Get Ready

Primary Trigonometric Ratios

The **primary trigonometric ratios** for $\angle A$ in right $\triangle ABC$ are

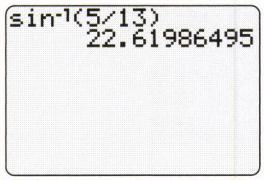
$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos A = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan A = \frac{\text{opposite}}{\text{adjacent}}$$

$$= \frac{5}{13} \qquad = \frac{12}{13} \qquad = \frac{5}{12}$$



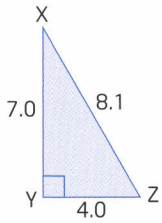
You can use the inverse trigonometric operations on a calculator to find angle measures. For example,

$$\angle A = \sin^{-1}\left(\frac{5}{13}\right)$$
$$\doteq 23°$$

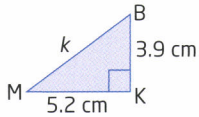


1. a) Find the primary trigonometric ratios for $\angle X$ in $\triangle XYZ$.

b) Find the measures of the acute angles in $\triangle XYZ$, to the nearest degree.



2. a) Apply the Pythagorean theorem to find side k. Then, find the primary trigonometric ratios for $\angle M$.

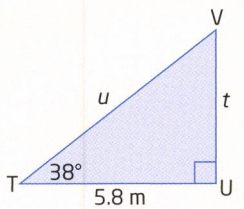


b) Find the measures of the acute angles in $\triangle BKM$, to the nearest degree.

Solve Triangles

To solve a triangle means to find all the unknown side lengths and unknown angle measures.

You can apply the primary trigonometric ratios to find the missing side lengths in right $\triangle TUV$.



$$\cos T = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan T = \frac{\text{opposite}}{\text{adjacent}}$$

$$\cos 38° = \frac{5.8}{u} \qquad \tan 38° = \frac{t}{5.8}$$

$$u = \frac{5.8}{\cos 38°} \qquad 5.8(\tan 38°) = t$$

$$u \doteq 7.4 \qquad 4.5 \doteq t$$

$$\angle V = 180° - 38° - 90°$$
$$= 52°$$

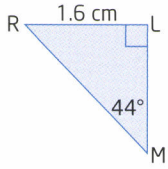
In $\triangle TUV$, u is 7.4 cm, t is 4.5 cm, and $\angle V$ is 52°.

Literacy Connections

When naming parts of triangles, use capital letters to represent the angles and vertices, and corresponding lowercase letters to represent opposite sides.

3. Describe two other methods that you could use to find side t in $\triangle TUV$.

4. Solve $\triangle RML$. Round side lengths to the nearest tenth of a centimetre.



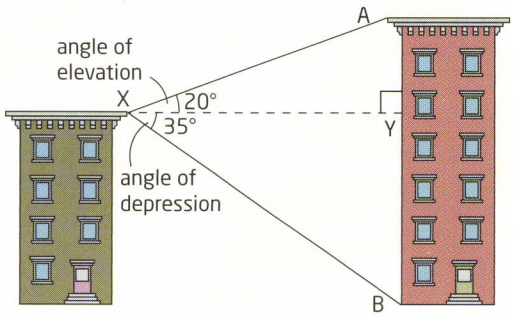
5. In $\triangle PWF$, $\angle P = 90°$, w is 4.8 km, and f is 5.3 km.

a) Draw and label $\triangle PWF$.

b) Solve $\triangle PWF$. Round side lengths to the nearest tenth of a kilometre and angle measures to the nearest degree.

Angles of Elevation and Depression

Angles of elevation and depression are measured above and below a horizontal line, respectively. For example, the angle of elevation from the top of one building to the top of a taller building is 20° and the angle of depression from the top of the shorter building to the base of the taller building is 35°.



6. The two buildings in the diagram above are 25 m apart.

a) Apply the tangent ratio to find BY, to the nearest tenth of a metre.

b) Use trigonometry to find the height of the taller building, to the nearest tenth of a metre.

7. From the top of a vertical cliff that is 40 m high, a ship is spotted at an angle of depression of 25°. How far from the base of the cliff is the ship, to the nearest metre?

Rearranging Formulas

Formulas are equations that show how two or more variables are related. You can rearrange a formula to isolate any of the variables in the equation. For example, $y = mx + b$ can be rearranged to express x in terms of y, m, and b.

$$y = mx + b$$

$$y - b = mx + b - b$$ Subtract b from both sides to isolate the term containing x.

$$y - b = mx$$

$$\frac{y - b}{m} = \frac{mx}{m}$$ Divide both sides by m to isolate x.

$$\frac{y - b}{m} = x$$

8. Rearrange each formula to isolate the variable indicated.

a) $y = mx + b$ for b

b) $s = \frac{d}{t}$ for d

c) $P = 4s$ for s

d) $P = a + b + c$ for b

9. Rearrange each formula to isolate the term or variable indicated.

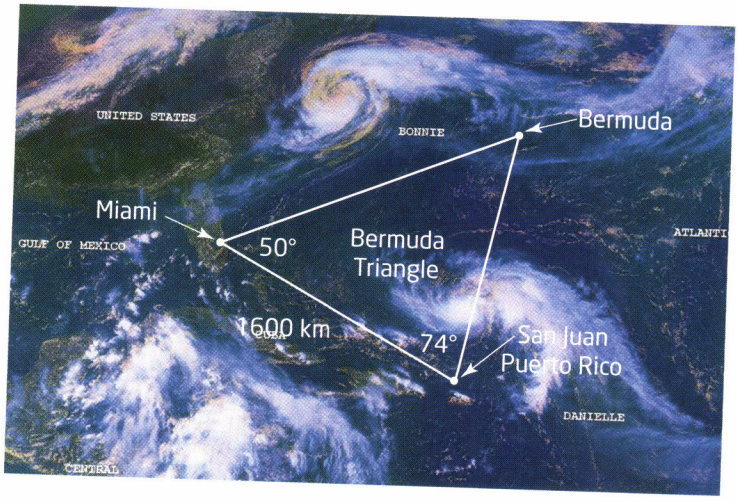
a) $a(\sin B) = b(\sin A)$ for a

b) $c^2 = a^2 + b^2$ for b^2

c) $A = 6s^2$ for s

d) $\frac{x}{\sin X} = \frac{y}{\sin Y}$ for y

8.1 The Sine Law



The Bermuda Triangle, in the north Atlantic Ocean, is the location of several unexplained plane and ship disappearances. Various theories have been suggested to explain this mysterious phenomenon, such as volatile weather patterns, magnetic compass variation, human error, and even alien abduction.

Suppose you took a cruise around the Bermuda Triangle. How could you determine its perimeter from the given information?

Investigate

Tools

- protractor and ruler

OR

- computer with *The Geometer's Sketchpad®*

Technology Tip

If you are using *The Geometer's Sketchpad®*, you can access the sine function from the **Functions** drop-down menu on the calculator.

Technology Tip

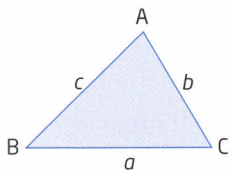
Instead of constructing new triangles to compare results, simply move one of the vertices of △ABC to examine other acute triangles. The calculated measures will change automatically.

How are the side lengths and sines of angles related in an acute triangle?

1. a) Draw an acute △ABC.

b) Measure the side lengths and the angles.

c) Calculate the sine of each angle.



2. a) Calculate the ratios $\frac{a}{b}$, $\frac{\sin A}{\sin B}$, $\frac{a}{c}$, $\frac{\sin A}{\sin C}$, $\frac{b}{c}$, and $\frac{\sin B}{\sin C}$.

b) Do you notice any relationships between the ratios? Explain.

3. a) Calculate each ratio.

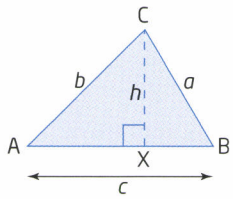
$$\frac{a}{\sin A} \qquad \frac{b}{\sin B} \qquad \frac{c}{\sin C}$$

b) Compare these results and explain what you notice.

4. Repeat steps 1 to 3 for two different triangles. Are the results the same? Explain.

5. Reflect Summarize the relationship of the sides and sines of angles in an acute triangle.

The relationship you investigated is called the **sine law**. To show why the sine law holds true, draw an acute triangle and add an altitude, h, from one of the vertices.



sine law

- the relationship between the sides and their opposite angles in any acute △ABC:
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The altitude splits △ABC into two smaller right triangles, △AXC and △BXC.

Find an expression for h.

Focus on △AXC:

$$\sin A = \frac{h}{b}$$

Multiply both sides by b.

$$b(\sin A) = h$$

Focus on △BXC:

$$\sin B = \frac{h}{a}$$

Multiply both sides by a.

$$a(\sin B) = h$$

Set the two expressions for h equal.

$$b(\sin A) = a(\sin B)$$

$$\frac{b(\cancel{\sin A})}{\cancel{\sin A}} = \frac{a(\sin B)}{\sin A}$$

Divide both sides by sin A.

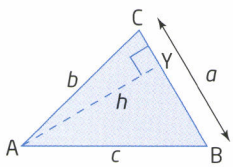
$$b = \frac{a(\sin B)}{\sin A}$$

$$\frac{b}{\sin B} = \frac{a(\cancel{\sin B})}{\sin A(\cancel{\sin B})}$$

Divide both sides by sin B.

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

This process can be repeated using a different altitude.



Combining the results gives the sine law.

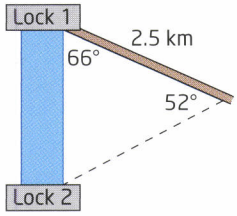
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Even though there are three parts to this equation, you only use two parts at a time. The choice of which two to use depends on what information is given.

You can apply the sine law to find an unknown side length of an acute triangle if you know two angles and one of the side lengths.

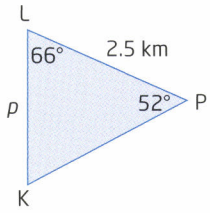
Example 1 Find a Side Length Using the Sine Law

A bicycle path forms a 66° angle with one lock of a canal. At a distance of 2.5 km along the bicycle path, the angle separating this lock from the next lock is 52°. How far apart are the two locks, to the nearest tenth of a kilometre?



Solution

Draw a simplified diagram showing the given information.



Before you can use the sine law to find length p, you need to find the measure of $\angle K$.

$\angle K = 180° - 66° - 52°$ The sum of the interior angles in a triangle is 180°.
$= 62°$

Now, apply the sine law.

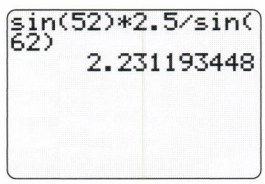
$$\frac{p}{\sin P} = \frac{k}{\sin K}$$

$$\frac{p}{\sin 52°} = \frac{2.5}{\sin 62°}$$ Substitute the given information.

$$\cancel{\sin 52°}\left(\frac{p}{\cancel{\sin 52°}}\right) = \sin 52°\left(\frac{2.5}{\sin 62°}\right)$$ Multiply both sides by sin 52°.

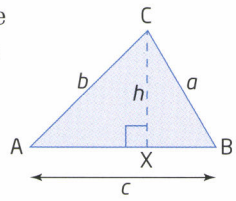
$$p = \sin 52°\left(\frac{2.5}{\sin 62°}\right)$$

$$p \doteq 2.2$$



The distance between the two locks is 2.2 km, to the nearest tenth of a kilometre.

You can also use the sine law to find an unknown angle if you know two sides and an angle opposite one of the known sides.



Sometimes the sine law appears in a different form. Recall the result obtained earlier for $\triangle ABC$.

$$b(\sin A) = a(\sin B)$$

$$\frac{b(\sin A)}{ab} = \frac{a(\sin B)}{ab}$$ Divide both sides by ab.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

Combining the results gives an alternative form of the sine law.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Either form can be used to find a missing side or angle—this alternative form is a little easier to use when finding an unknown angle.

Literacy Connections

A proportion is a statement that says that two ratios are equal. For example, $\frac{2}{3} = \frac{4}{6}$

The proportion still holds true if you invert the numerator and denominator of *both* ratios. $\frac{3}{2} = \frac{6}{4}$

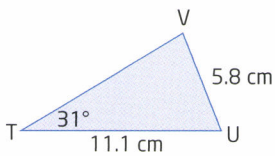
This is why either form of the sine law can be applied.

Example 2 Find an Angle Using the Sine Law

In acute $\triangle TUV$, TU = 11.1 cm, UV = 5.8 cm, and $\angle T = 31°$. Find the measure of $\angle V$, to the nearest degree.

Solution

Draw a diagram and label the given information.



Apply the sine law.

$$\frac{\sin V}{v} = \frac{\sin T}{t}$$

$$\frac{\sin V}{11.1} = \frac{\sin 31°}{5.8}$$ Substitute the given information.

$$11.1\left(\frac{\sin V}{11.1}\right) = 11.1\left(\frac{\sin 31°}{5.8}\right)$$ Multiply both sides by 11.1.

$$\sin V = 11.1\left(\frac{\sin 31°}{5.8}\right)$$

$$\sin V = 0.9856\ldots$$

$$\angle V = \sin^{-1}(0.9856\ldots)$$ Use the inverse sine operation.

$$\angle V \doteq 80°$$

```
11.1*sin(31)/5.8
        .9856763158
sin⁻¹(Ans)
        80.29077071
```

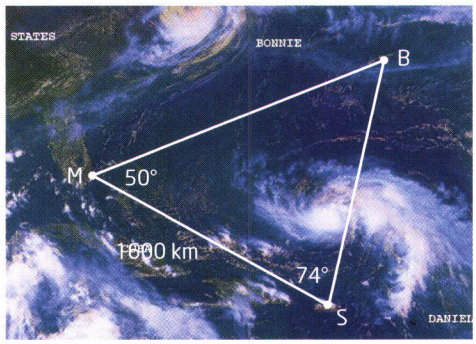
The measure of $\angle V$ is about 80°.

Technology Tip

Ans in the calculator display indicates the last answer. To use the last answer in a calculation, press 2nd (−) for [ANS].

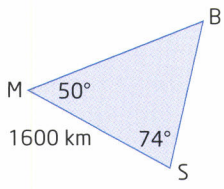
Example 3 Perimeter of the Bermuda Triangle

Use the information given on the diagram to determine the perimeter of the Bermuda Triangle, to the nearest hundred kilometres.



Solution

Draw a simplified diagram showing the given information.



First, find $\angle B$.

$$\begin{aligned}\angle B &= 180° - 50° - 74° \\ &= 56°\end{aligned}$$

Now, apply the sine law twice to find sides m and s.

$$\frac{m}{\sin M} = \frac{b}{\sin B}$$

$$\frac{m}{\sin 50°} = \frac{1600}{\sin 56°}$$

$$m = \sin 50° \left(\frac{1600}{\sin 56°}\right)$$

$$m \doteq 1478$$

Method 1: Use Side *b*

$$\frac{s}{\sin S} = \frac{b}{\sin B}$$

$$\frac{s}{\sin 74°} = \frac{1600}{\sin 56°}$$

$$s = \sin 74° \left(\frac{1600}{\sin 56°}\right)$$

$$s \doteq 1855$$

Method 2: Use Side *m*

$$\frac{s}{\sin S} = \frac{m}{\sin M}$$

$$\frac{s}{\sin 74°} = \frac{1478}{\sin 50°}$$

$$s = \sin 74° \left(\frac{1478}{\sin 50°}\right)$$

$$s \doteq 1855$$

To find the perimeter, add the sides.

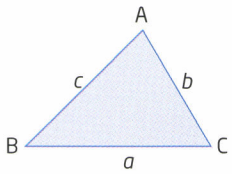
$$\begin{aligned}P &= b + m + s \\ &= 1600 + 1478 + 1855 \\ &= 4933\end{aligned}$$

Therefore, the perimeter of the Bermuda Triangle is approximately 4900 km.

Key Concepts

- In an acute $\triangle ABC$, the sine law states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



- The sine law can be used to find
  - an unknown side if two angles and a side are known
  - an unknown angle if two sides and the angle opposite one of the known sides are known
- The sine law can also be written in the form

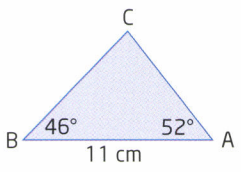
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

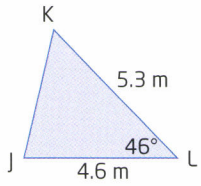
Communicate Your Understanding

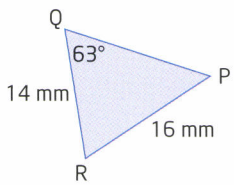
C1 **a)** Why does the sine law have three parts to its equation?

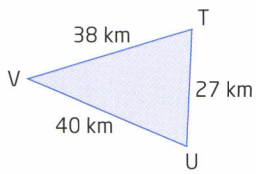
b) Do you use all three parts at once? How can you tell which parts to use?

C2 Which of the following triangles can be solved using the sine law? For those that can, explain how. For those that cannot, explain why not.









C3 How many pieces of information about a triangle's side lengths and angles are needed in order to solve it using the sine law? What possible combinations will work?

C4 There are two forms of the sine law, relating sides a and b and $\angle A$ and $\angle B$:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \text{ and } \frac{\sin A}{a} = \frac{\sin B}{b}$$

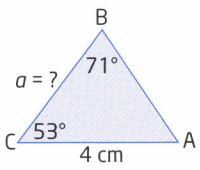
Explain when it is easier to apply each form. Create examples to support your explanation.

Practise

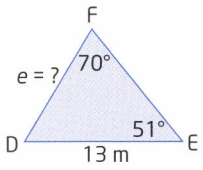
For help with questions 1 and 2, see Example 1.

1. Find the length of the indicated side in each triangle, to the nearest unit.

a)

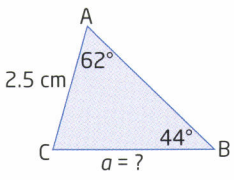


b)

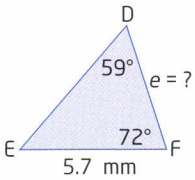


2. Find the length of the indicated side in each triangle, to the nearest tenth of a unit.

a)



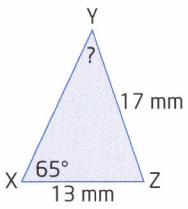
b)



For help with questions 3 and 4, see Example 2.

3. Find the measure of the indicated angle in each triangle, to the nearest degree.

a)



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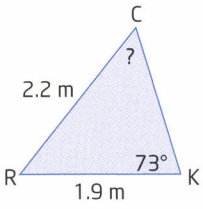


4. Draw a diagram and label the given information. Then, find the measure of the indicated angle in each triangle, to the nearest degree.

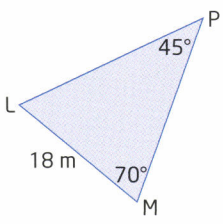
a) In acute $\triangle GHK$, $\angle G = 47°$, $h = 5$ cm, and $g = 4$ cm. Find $\angle H$.

b) In acute $\triangle RST$, $\angle S = 72°$, $t = 1.5$ m, and $s = 1.8$ m. Find $\angle T$.

For help with questions 5 to 7, see Example 3.

5. Solve each triangle. Round answers to the nearest unit, if necessary.

a)



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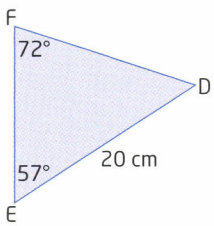
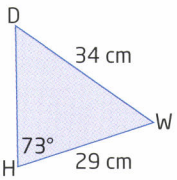
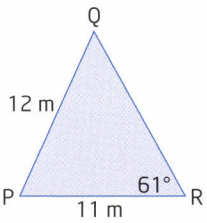


6. Solve each triangle. Round answers to the nearest unit, if necessary.

a)



b)



7. Draw a diagram and label the given information. Then, solve each triangle. Round answers to the nearest unit, if necessary.

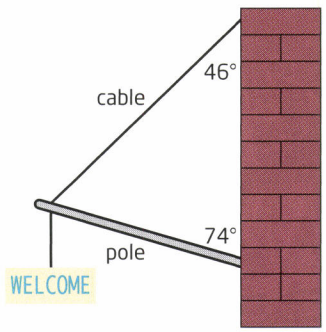
a) In acute $\triangle AKR$, $k = 15$ mm, $r = 13$ mm, and $\angle K = 68°$.

b) In acute $\triangle UJF$, $j = 23$ km, $\angle U = 57°$, and $\angle F = 48°$.

8. **Use Technology** Check your answers to question 7 using dynamic geometry software.

Connect and Apply

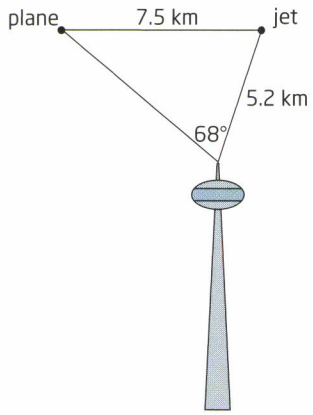
9. A sign is supported by a pole and a cable, as shown. The cable is attached to the wall 2.2 m above the base of the pole.



a) Find the length of the pole, to the nearest tenth of a metre.

b) Find the length of the cable, to the nearest tenth of a metre.

10. A small commercial plane and a jet airliner are 7.5 km from each other, at the same altitude. From an observation tower, the two aircraft are separated by an angle of 68°. If the jet airliner is 5.2 km from the observation tower, how far is the commercial plane from the observation tower, to the nearest tenth of a kilometre?



11. A telephone pole is supported by an 18-m guy wire that makes an angle of 50° with the horizontal ground. A 14-m guy wire is to be fastened on the other side of the pole for reinforcement. Both wires attach to the pole at its top. Round your answers to the nearest unit, if necessary.

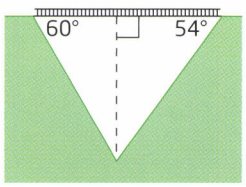
a) What angle should the second wire make with the ground?

b) How tall is the pole?

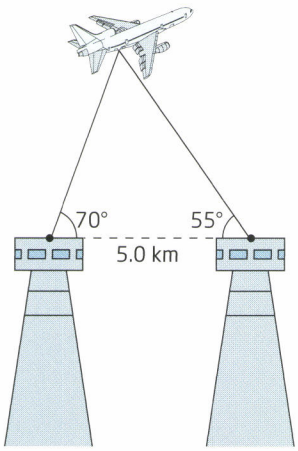
c) How far is the base of each wire from the base of the pole?

d) Could you solve this problem without using the sine law? Explain.

12. A bridge across a valley is 150 m in length. The valley walls make angles of 60° and 54° with the bridge that spans it, as shown. How deep is the valley, to the nearest metre?

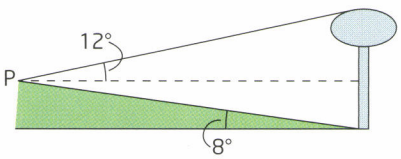


13. Chapter Problem You and your partner are observing an aircraft from two observation decks, located 5.0 km apart. From your point of view, the aircraft is at an angle of elevation of 70°. From your partner's point of view, the angle of elevation is 55°. Determine the altitude of the aircraft, to the nearest tenth of a kilometre.



14. In isosceles $\triangle ABC$, $c = 15$ cm, $a = 11$ cm, and $\angle B = 68.5°$. Can this triangle be solved using the sine law? If so, solve it and explain your reasoning. If not, explain why not.

15. To measure the height of a water tower, Kelly walks 0.15 km from the base of the tower along an inclined path to point P. From P, the top of the tower appears at an angle of elevation of 12°.



Find the height of the water tower, to the nearest metre.

16. You can find the area of a triangle if you know all three of its side lengths by applying Heron's formula, $A = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b, and c are the side lengths, and $s = \frac{1}{2}(a+b+c)$.

a) Use Heron's formula to find the area of the Bermuda Triangle.

b) Check this result by estimating the height of the triangle and applying the formula for the area of a triangle, $A = \frac{bh}{2}$.

Did You Know?

Heron was a mathematician, engineer, and inventor born in Alexandria, Greece, during the first century. Among his accomplishments was the invention of the first steam engine, called an *aelopile*. The principle Heron used in the aelopile is similar to that used in today's jet engines.

17. Refer to question 16. Lan's garden is in the shape of an acute triangle. Two of the vertices have angles of 55° and 58°. The side joining these vertices is 6.2 m in length. What is the area of this garden, to the nearest tenth of a square metre?

18. a) Create a problem involving the sine law for which the answer is 15 m.

b) Solve the problem.

c) Trade with a partner and solve each other's problem. Check your solutions.

19. Does the sine law work if you replace sines with cosines or tangents? Investigate several triangles and explain whether there is such a relationship.

Extend

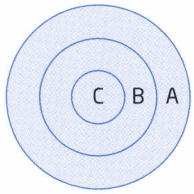
20. a) Show that the sine law simplifies to the sine ratio when one of the angles of a triangle is 90°.

b) Would you use the sine law to solve right triangles? Explain.

21. Refer to question 16. Show that Heron's formula can also be written as follows:

$$A = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}$$

22. Math Contest Frishta, Michelle, Daniel, and Vahpav each shoot two arrows at the target shown. Each ring has a different point value.



- Frishta shoots one arrow in ring B and one in ring C. Her score is 17.
- Michelle shoots one in ring A and one in ring B. Her score is 12.
- Daniel shoots one in ring C and one in ring A. His score is 15.
- Vahpav shoots two arrows into ring A.

Vahpav's score is

A 8

B 10

C 12

D 18

E 20

23. Math Contest When each of three different numbers is added to the average of the other two, the results are 105, 106, and 125. What is the average of the three original numbers?

A 51

B 55

C 56

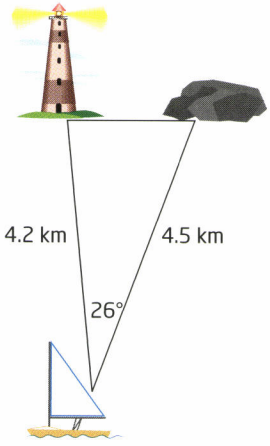
D 57

E 63

8.2 The Cosine Law

Sailing can be a lot of fun as long as you are careful to avoid hazardous situations, such as shallow water. Occasionally, when the ship's navigational instruments are out of order, a ship's captain can use trigonometry to find important distances and directions.

Suppose that as captain you know the measures of two sides and one angle in the acute triangle shown. Can you use the sine law to solve the triangle?



Tools

- protractor and ruler

OR

- computer with *The Geometer's Sketchpad®*

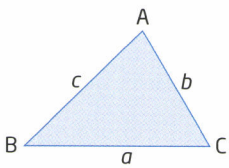
Technology Tip

If you are using *The Geometer's Sketchpad®*, calculate the square of a measure by clicking **^2** after it.

To calculate $2ab(\cos C)$, use the * key for multiplication and access the cosine function from the **Functions** drop-down menu on the calculator.

Investigate

How are the side lengths and cosines of angles related in an acute triangle?



1. a) Draw an acute $\triangle ABC$.

b) Measure the side lengths a, b, and c.

c) Measure $\angle C$.

2. a) Calculate a^2, b^2, and c^2.

b) For acute $\triangle ABC$, compare $a^2 + b^2$ and c^2. When would $a^2 + b^2 = c^2$? What is this relationship called?

c) Calculate $2ab(\cos C)$.

d) Describe any relationship you notice between $a^2 + b^2$, c^2, and $2ab(\cos C)$.

e) Examine two or three other acute triangles to see if this relationship holds true.

3. Suppose $\angle C$ is 90°.

a) What happens to the quantity $2ab(\cos C)$?

b) What relationship applies to the three sides of the triangle when this happens?

4. Reflect

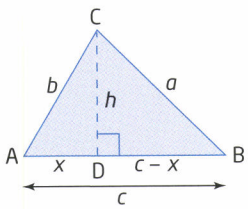
a) Write an equation that relates the cosine of an angle and the three sides of an acute triangle.

b) What happens to this relationship when the measure of one of the angles is 90°?

cosine law

- the relationship between the cosine of an angle and the lengths of the three sides in any acute △ABC:
  $a^2 = b^2 + c^2 - 2bc(\cos A)$
  $b^2 = c^2 + a^2 - 2ca(\cos B)$
  $c^2 = a^2 + b^2 - 2ab(\cos C)$

The **cosine law** relates the cosine of an angle to the three side lengths of an acute triangle. To derive the cosine law, draw a triangle and add an altitude, h, from one of the vertices.



The altitude splits △ABC into two smaller right triangles, △ADC and △BDC.

Let AD = x.
Then, BD = $c - x$.

Focus on △ADC:
From the Pythagorean theorem, $b^2 = x^2 + h^2$.
Also, the cosine ratio gives

$$\frac{x}{b} = \cos A$$

$$x = b(\cos A)$$

Focus on △BDC:
Write an equation using the Pythagorean theorem.

$$\begin{aligned} a^2 &= h^2 + (c - x)^2 \\ &= h^2 + c^2 - 2cx + x^2 && \text{Expand the binomial.} \\ &= x^2 + h^2 + c^2 - 2cx && \text{Rearrange the terms.} \\ &= b^2 + c^2 - 2c[b(\cos A)] && \text{Substitute } b^2 \text{ for } x^2 + h^2 \text{ and } b(\cos A) \text{ for } x. \\ &= b^2 + c^2 - 2cb(\cos A) \\ &= b^2 + c^2 - 2bc(\cos A) \end{aligned}$$

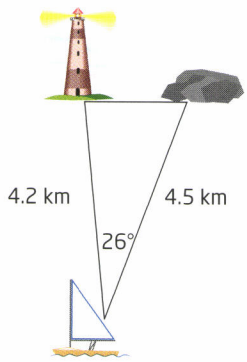
This equation allows you to find the side length a if you know the side lengths b and c and the measure of $\angle A$.

You can derive similar equations for the other side lengths. Combining the results gives the three forms of the cosine law.

$$a^2 = b^2 + c^2 - 2bc(\cos A)$$
$$b^2 = c^2 + a^2 - 2ca(\cos B)$$
$$c^2 = a^2 + b^2 - 2ab(\cos C)$$

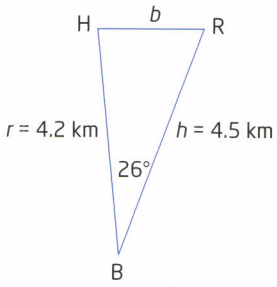
Example 1 Find a Side Length Using the Cosine Law

A boat is sailing north through a narrow strait. Through one particularly narrow section, a lighthouse marks the western shoreline, while a buoy indicates a rock hazard directly east of the lighthouse, as shown. What is the width of this section of the strait, to the nearest tenth of a kilometre?



Solution

Draw a simplified diagram and label the given information. Choose variables for the vertices.



Since two sides and the contained angle are given, use the cosine law to find b.

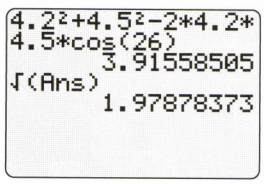
$b^2 = r^2 + h^2 - 2rh(\cos B)$

$b^2 = 4.2^2 + 4.5^2 - 2(4.2)(4.5)(\cos 26°)$ **Substitute the given information.**

$b^2 = 3.9155...$

$b = \sqrt{3.9155...}$ **Take the square root of both sides.**

$b \doteq 2.0$



The width of this section of the strait is approximately 2.0 km.

Did You Know?

The Clipper Round The World Yacht Race gives paying amateur crew members the chance to sail around the world. The race started in 1996 and takes place every 2 years.

It is the world's longest yacht race, at over 35 000 nautical miles, and takes the fleet of identical yachts 10 months to complete.

Literacy Connections

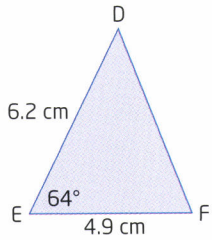
A contained angle is the interior angle that is formed at the vertex of two adjacent sides in a triangle.

Example 2 Solve a Triangle

In acute $\triangle DEF$, $d = 4.9$ cm, $f = 6.2$ cm, and $\angle E = 64°$. Solve $\triangle DEF$. Round measures to the nearest degree or tenth of a centimetre, if necessary.

Solution

Draw a diagram and label the given information.



Since two sides and the contained angle are given, use the cosine law first to find side e.

Since I do not know the length of the side opposite the given angle, $\angle E$, I cannot use the sine law.

$e^2 = d^2 + f^2 - 2df(\cos E)$

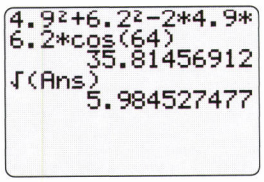
$e^2 = 4.9^2 + 6.2^2 - 2(4.9)(6.2)(\cos 64°)$

$e^2 = 35.814...$

$e = \sqrt{35.814...}$

$e \doteq 6.0$

Side e is about 6.0 cm.

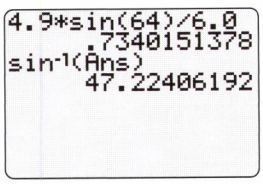


Use the sine law to find the measure of one of the other angles.

$\dfrac{\sin D}{d} = \dfrac{\sin E}{e}$

$\dfrac{\sin D}{4.9} = \dfrac{\sin 64°}{6.0}$

$\sin D = 4.9\left(\dfrac{\sin 64°}{6.0}\right)$



$\sin D = 0.7340...$

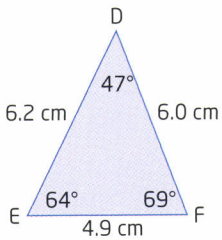
$\angle D = \sin^{-1}(0.7340...)$

$\angle D \doteq 47°$

Determine the measure of the third angle.

$\angle F = 180° - 47° - 64°$

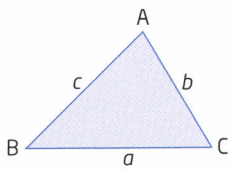
$= 69°$



$\triangle DEF$ has been solved for all side lengths and angle measures.

Key Concepts

- In an acute $\triangle ABC$, the cosine law states that
  $a^2 = b^2 + c^2 - 2bc(\cos A)$
  $b^2 = c^2 + a^2 - 2ca(\cos B)$
  $c^2 = a^2 + b^2 - 2ab(\cos C)$



- The cosine law can be used to find a missing side if the other two sides and their contained angle are known.

Communicate Your Understanding

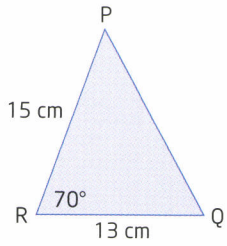
C1 **a)** What is meant by a contained angle in a triangle? Draw a diagram to illustrate your answer.

b) Why is this concept important when applying the cosine law?

C2 **a)** How are the cosine law and the Pythagorean theorem similar? How are they different?

b) Under what conditions can you apply
- the Pythagorean theorem?
- the cosine law?

C3 Describe the steps you would use to solve $\triangle PQR$.

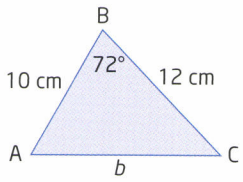


Practise

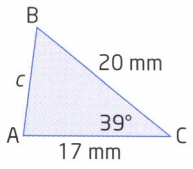
For help with questions 1 to 3, see Example 1.

1. Find the missing side length in each triangle, to the nearest unit.

a)

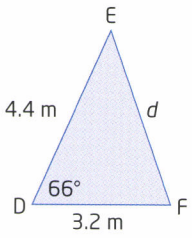


b)



2. Find the missing side length in each triangle, to the nearest tenth of a unit.

a)



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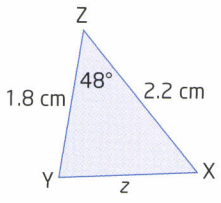
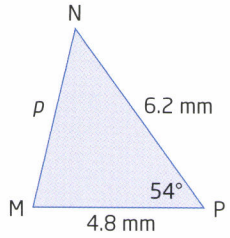


c)



3. Sketch each triangle and use the given information to find the missing side length, to the nearest tenth of a unit.

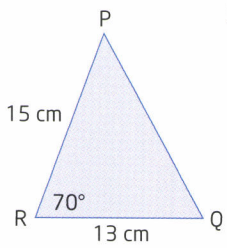
a) In acute $\triangle TUV$, $t = 1.8$ cm, $v = 1.4$ cm, and $\angle U = 52°$.

b) In acute $\triangle DEF$, $e = 1.1$ km, $f = 1.6$ km, and $\angle D = 74°$.

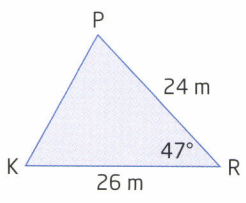
For help with questions 4 and 5, see Example 2.

4. Solve each triangle. Round answers to the nearest unit, if necessary.

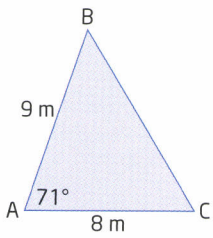
a)



b)



c)



5. Draw a diagram and label the given information. Then, solve each triangle. Round answers to the nearest unit, if necessary.

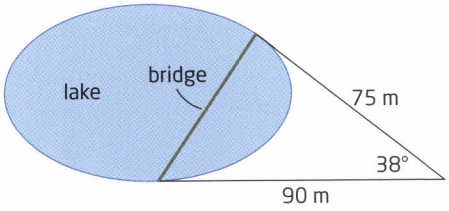
a) In acute $\triangle EFG$, $e = 5$ cm, $f = 6$ cm, and $\angle G = 63°$.

b) In acute $\triangle WXY$, $w = 10$ m, $y = 11$ m, and $\angle X = 80°$.

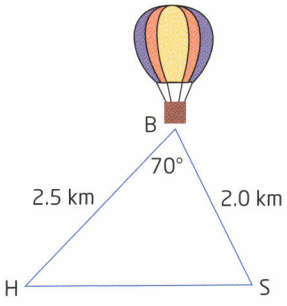
6. **Use Technology** Check your answers to question 5 using dynamic geometry software.

Connect and Apply

7. Find the length of the bridge, to the nearest metre.



8. Chandra is riding in a hot-air balloon and spots her house and her school. She estimates how far away they are from her, and the angle separating their lines of sight, as shown.

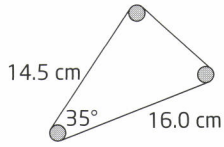


Use Chandra's estimated measures.

a) How far apart are Chandra's home and school, to the nearest tenth of a kilometre?

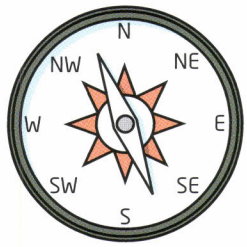
b) Chandra's mom is watching her from home, and her friends are watching from school. At what angle of elevation does Chandra appear to each of them, to the nearest degree?

9. A drive belt wraps around three pulleys, as shown.



Find the total length of the drive belt, to the nearest tenth of a centimetre. Ignore the curved sections.

10. Two ships leave port at the same time, travelling at the same speed of 10 knots. *Wavedancer* sails north, while *Ocean Princess* travels northeast.



a) How far apart are the two ships after 1 h?

b) How far apart are they after 2 h?

c) How do these answers change if *Wavedancer* travels twice as fast? Justify your answer.

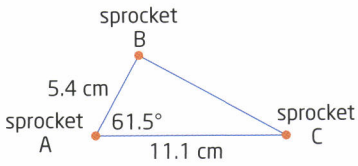
Did You Know?

A "knot" is short for a "nautical mile per hour."

1 knot = 1.852 km/h

1 nautical mile is the distance between two points on the equator that are separated by $\frac{1}{60}$ of a degree, making the circumference of Earth 21 600 nautical miles.

11. Connor is building a toy model of a track-type bulldozer. Three sprockets for one of the tracks are to be assembled as shown.



a) How far should sprocket C be placed from sprocket B, to the nearest tenth of a centimetre?

b) Find the interior angles of the triangle formed by these sprockets, to the nearest tenth of a degree.

12. **a)** Create a problem involving the cosine law for which the answer is 24 cm.

b) Solve the problem.

c) Trade with a partner and solve each other's problem. Check your solutions.

13. The airport at Goderich, Ontario, has two runways with lengths 1525 m and 915 m. The beginnings of the runways meet at an angle of 37°. The other ends of the runways are called the thresholds.

a) Draw a diagram and label the given information.

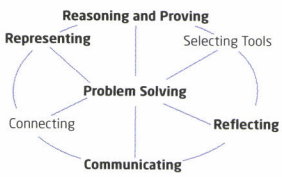
b) How far apart are the thresholds of the runways, to the nearest metre?

14. **Chapter Problem** While cruising at a steady speed of 400 km/h, you identify a storm cloud straight ahead 45 km away. To avoid turbulence, you start climbing at an angle of elevation of 15°. If you maintain this speed and direction for 6 min, how far will you be from the storm cloud? Round to the nearest kilometre.

15. Show that the cosine law simplifies to the Pythagorean theorem when the contained angle between the two known sides is 90°.

Achievement Check

16. One of the tallest totem poles in the world is located in Alert Bay, British Columbia. When the angle of elevation of the Sun is 62°, the totem pole casts a shadow of 30 m.



a) Suppose the totem stood vertical—how tall would it be, to the nearest tenth of a metre?

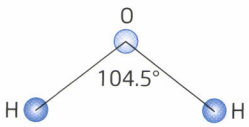
b) Suppose it was not quite vertical so that it made an angle of 89° with the ground. In this case, would your answer for its height be taller or shorter than your answer in part a)? Justify without calculations.

c) Calculate the difference between these two heights.

Extend

17. Design and carry out an investigation to determine if the cosine law holds true for obtuse triangles. Write a brief report of your findings.

18. Lee is building a scale model of a water molecule for his science project. The molecule consists of one oxygen atom and two hydrogen atoms, chemically bonded as shown.



Lee models the bond for each hydrogen atom with the oxygen atom using a 10-cm straw.

a) How far will the two hydrogen atoms be from each other, to the nearest tenth of a centimetre?

b) What angles will a line joining the two hydrogen atoms make with the lines of their chemical bonds?

19. Refer to question 9. Suppose that the measured distances are taken between the centres of each pulley, and that the diameter of each pulley is 2.8 cm. Find a more accurate total length of the drive belt. State any assumptions you make.

20. **Math Contest** Sam and Nick are both members of the basketball team, along with 10 other players. When the team is seated on the bench, what is the probability that Sam and Nick do not sit beside each other?

A $\frac{11}{12}$ **B** $\frac{1}{6}$

C $\frac{3}{4}$ **D** $\frac{5}{6}$

E $\frac{10}{11}$

21. **Math Contest** Show that for any $\angle A$, $(\sin A)^2 + (\cos A)^2 = 1$.

8.3

Find Angles Using the Cosine Law

You have learned when the sine law and cosine law can each be used to solve acute triangles.

What if you are given three side lengths in an acute triangle?

Runners in a 10-K race follow a triangular course. The three legs of the race, in order, are 3.8 km, 2.4 km, and 3.8 km. Can you apply the sine law or the cosine law to find the angle between the starting leg and the finishing leg?

Investigate

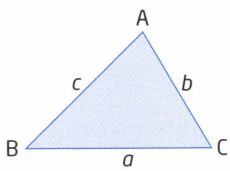
Tools

- protractor
- ruler

How can you solve an acute triangle if three side lengths are given?

1. a) Draw an acute scalene $\triangle ABC$.

b) Measure the side lengths a, b, and c.



2. Can you use the lengths of the three sides with the sine law to find the measure of one of the angles in $\triangle ABC$? Explain.

3. a) Can you use the lengths of the three sides with the cosine law to find the measure of one of the angles in $\triangle ABC$? Explain.

b) Does the choice of angle matter? Explain.

4. a) Solve for one of the unknown angles using a trigonometric law and algebraic manipulation.

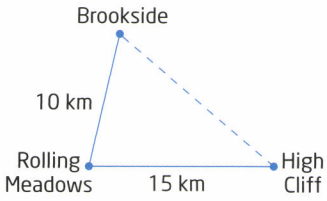
b) Check your answer with a protractor.

5. Reflect Explain which trigonometric law can be used to find an angle measure if three side lengths are known in an acute triangle. How would you solve the equation?

If you know any three measurements of an acute triangle, including at least one side length, you can find the remaining measurements using the sine law and the cosine law. The choice depends on which law allows you to relate the three given measurements to one unknown measurement.

Example 1 Find an Angle Using the Cosine Law

Three towns are connected by two roads, as shown.



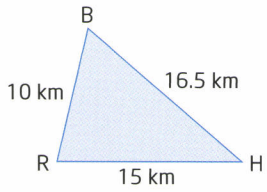
A third road is planned that will directly connect Brookside and High Cliff, which are 16.5 km apart. Find the angle, to the nearest tenth of a degree, between the new road and the existing road from

a) Rolling Meadows to Brookside

b) Rolling Meadows to High Cliff

Solution

Draw a simplified diagram and label it with the given information.

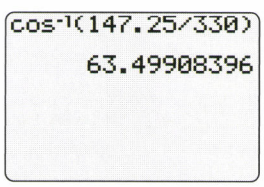


The new road will make unknown angles at B and H.

a) Use the cosine law to find $\angle B$.

Method 1: Substitute, Then Rearrange

$$b^2 = h^2 + r^2 - 2hr(\cos B)$$
$$15^2 = 10^2 + 16.5^2 - 2(10)(16.5)(\cos B)$$
$$225 = 100 + 272.25 - 330(\cos B) \quad \text{Simplify.}$$
$$225 = 372.25 - 330(\cos B)$$
$$225 - 372.25 = -330(\cos B) \quad \text{Isolate the term containing cos B.}$$
$$-147.25 = -330(\cos B)$$
$$\frac{-147.25}{-330} = \frac{-330(\cos B)}{-330} \quad \text{Divide both sides by -330.}$$
$$\frac{147.25}{330} = \cos B$$
$$\cos^{-1}\left(\frac{147.25}{330}\right) = \angle B \quad \text{Apply the inverse cosine to find } \angle B.$$
$$63.5° \doteq \angle B$$



The new road will make an angle of approximately 63.5° with the existing road from Rolling Meadows to Brookside.

Method 2: Rearrange, Then Substitute

$$b^2 = h^2 + r^2 - 2hr(\cos \text{B})$$

$$b^2 - h^2 - r^2 = -2hr(\cos \text{B})$$ Isolate the term containing cos B.

$$\frac{b^2 - h^2 - r^2}{-2hr} = \frac{-2hr(\cos \text{B})}{-2hr}$$ Divide both sides by $-2hr$.

$$\frac{b^2 - h^2 - r^2}{-2hr} = \cos \text{B}$$

$$\frac{15^2 - 10^2 - 16.5^2}{-2(10)(16.5)} = \cos \text{B}$$ Substitute the known information.

$$\frac{-147.25}{-330} = \cos \text{B}$$ Simplify.

$$\cos^{-1}\left(\frac{147.25}{330}\right) = \angle \text{B}$$ Apply the inverse cosine to find $\angle$B.

$$63.5 \doteq \angle \text{B}$$

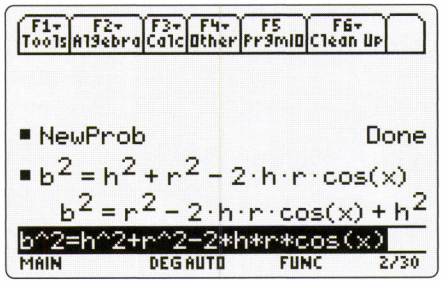
The new road will make an angle of approximately 63.5° with the existing road from Rolling Meadows to Brookside.

Method 3: Rearrange, Then Substitute, Using a Computer Algebra System

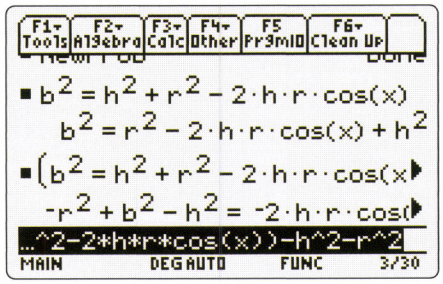
Ensure that angle measurements are calculated in degrees.

Let $\angle$B be represented by x.

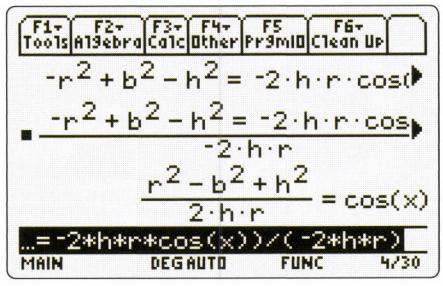
In the Home screen, type the cosine law:
$b^2 = h^2 + r^2 - 2hr(\cos x)$



Subtract h^2 and r^2 to isolate the term with cos x.



Divide by the expression $-2hr$.



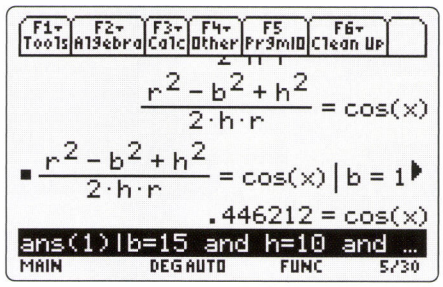
Technology Tip

To set the angle mode to degrees on a TI-89 calculator:

- Press MODE. Cursor down to the **Angle** mode setting.
- Press ▶, and select **2:DEGREE**.
- Press ENTER to save the settings.

Substitute the values for b, h, and r.

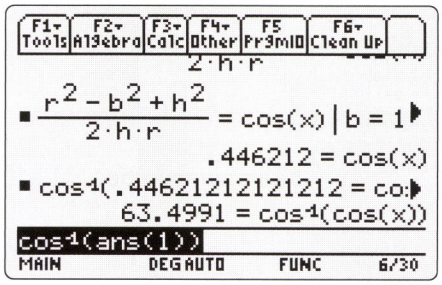
- Press [2nd] [(−)] for [ANS], followed by the *such that* symbol, [|].
- Type $b = 15$. Press [2nd] [MATH] to display the **MATH** menu, select **8:Test**, and then **8:and**.
- Type $h = 10$. Press [2nd] [MATH], select **8:Test**, and then **8:and**.
- Type $r = 16.5$ and press [ENTER].


F1 Tools
F2 Algebra
F3 Calc
F4 Other
F5 PrgmIO
F6 Clean Up
$\frac{r^2 - b^2 + h^2}{2 \cdot h \cdot r} = \cos(x)$
$\frac{r^2 - b^2 + h^2}{2 \cdot h \cdot r} = \cos(x) \mid b = 1$
.446212 = cos(x)
ans(1)|b=15 and h=10 and ...
MAIN DEG AUTO FUNC 5/30

Find the angle.

- Press [◆] [COS^{-1}], and then [2nd] [ANS], followed by [)].
- Press [ENTER].

The new road will make an angle of approximately 63.5° with the existing road from Rolling Meadows to Brookside.



b) You could apply the cosine law a second time to find ∠H. However, now you have enough information to apply the sine law. The sine law requires fewer calculation steps.

$$\frac{\sin H}{h} = \frac{\sin B}{b}$$

$$\frac{\sin H}{10} = \frac{\sin 63.5°}{15}$$

$$\sin H = 10\left(\frac{\sin 63.5°}{15}\right)$$

$$\sin H = 0.5966\ldots$$

$$\angle H = \sin^{-1}(0.5966\ldots)$$

$$\angle H \doteq 36.6°$$

The new road will make an angle of approximately 36.6° with the existing road from Rolling Meadows to High Cliff.

Example 2 Solve an Acute Triangle

Runners in a 10-K race follow a triangular course. The three legs of the race, in order, are 3.8 km, 2.4 km, and 3.8 km. Find the angle between the legs in each pair, to the nearest degree.

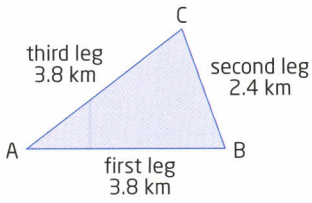
a) first and second

b) second and third

c) first and third

Solution

Draw a diagram and label it with the known information.



a) Since the measures of three sides are given, use the cosine law to find $\angle B$.

$$b^2 = a^2 + c^2 - 2ac(\cos B)$$

$$\cos B = \frac{b^2 - a^2 - c^2}{-2ac}$$ Rearrange to isolate cos B.

$$\cos B = \frac{3.8^2 - 2.4^2 - 3.8^2}{-2(2.4)(3.8)}$$ Substitute known values.

$$\cos B = \frac{-5.76}{-18.24}$$ Simplify.

$$\angle B = \cos^{-1}\left(\frac{5.76}{18.24}\right)$$ Solve for $\angle B$.

$$\angle B \doteq 72°$$

The angle between the first and second legs of the race is 72°.

b) Method 1: Apply the Sine Law

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\frac{\sin C}{3.8} = \frac{\sin 72°}{3.8}$$

$$\sin C = 3.8\left(\frac{\sin 72°}{3.8}\right)$$

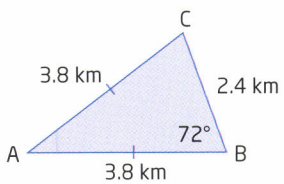
$$\sin C = \sin 72°$$

Therefore, $\angle C = 72°$.

The angle between the second and third legs of the race is 72°.

Method 2: Apply Geometric Reasoning

Since AC = AB, $\triangle ABC$ is isosceles.
The two angles opposite the two equal sides of an isosceles triangle are equal.
Since $c = b$, then $\angle C = \angle B$. Therefore, $\angle C = 72°$.
The angle between the second and third legs of the race is 72°.



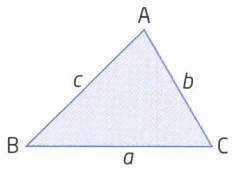
c) Use the two known angles to find $\angle A$.

$$\angle B = 180° - 72° - 72°$$
$$= 36°$$

The angle between the first and third legs of the race is 36°.

Key Concepts

- You can rearrange the cosine law to find an angle if you know three side lengths of an acute $\triangle ABC$.



For example, to find the measure of $\angle B$, rearrange the appropriate form of the cosine law.

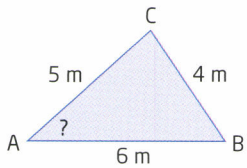
$$b^2 = a^2 + c^2 - 2ac(\cos B) \Rightarrow \cos B = \frac{b^2 - a^2 - c^2}{-2ac}$$

- Once you have found one angle, you can apply the sine law to find a second angle.
- When applying trigonometry to solve problems involving acute triangles, there is often more than one valid strategy.

Communicate Your Understanding

C1 When can you use the sine law to find an unknown angle in an acute triangle? When must you use the cosine law? Draw diagrams to support your answer.

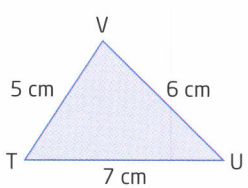
C2 The following steps show how the cosine law can be rearranged to find $\angle A$. Copy the steps and write a short explanation beside each one.



Steps	Explanation
$a^2 = b^2 + c^2 - 2bc(\cos A)$	Write the cosine law that includes $\angle A$.
$a^2 - b^2 - c^2 = -2bc(\cos A)$	
$\frac{a^2 - b^2 - c^2}{-2bc} = \frac{-2bc(\cos A)}{-2bc}$	
$\frac{a^2 - b^2 - c^2}{-2bc} = \cos A$	
$\frac{4^2 - 5^2 - 6^2}{-2(5)(6)} = \cos A$	
$\frac{-45}{-60} = \cos A$	
$\cos^{-1}\left(\frac{45}{60}\right) = \angle A$	
$41° \doteq \angle A$	

C3 **a)** Describe the steps you would take to solve $\triangle TUV$.

b) Describe a different set of steps that will also work.

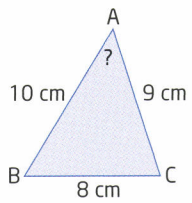


Practise

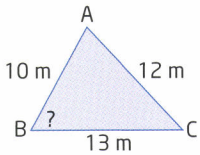
For help with questions 1 to 3, see Example 1.

1. Solve for the indicated angle, to the nearest degree.

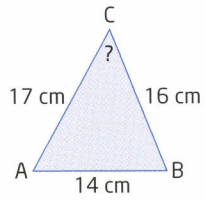
a)



b)

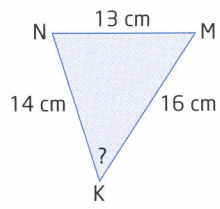


c)

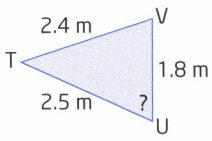


2. Solve for the indicated angle, to the nearest degree.

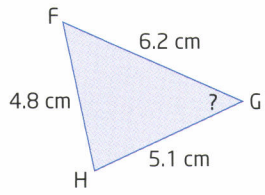
a)



b)



c)



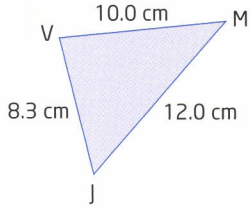
3. Sketch each triangle. Then, use the given information to find the indicated angle, to the nearest degree.

a) In acute $\triangle ARD$, $a = 170$ mm, $r = 190$ mm, and $d = 210$ mm. Find $\angle D$.

b) In acute $\triangle HWN$, $h = 1.4$ km, $w = 1.7$ km, and $n = 1.2$ km. Find $\angle W$.

For help with questions 4 to 6, see Example 2.

4. Consider $\triangle JVM$.



a) Follow these steps in order to solve $\triangle JVM$. Round answers to the nearest tenth of a degree.

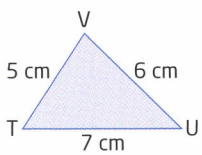
- Use the cosine law to find $\angle J$.
- Use the cosine law to find $\angle V$.
- Use the cosine law to find $\angle M$.

b) Solve $\triangle JVM$ using a more efficient method.

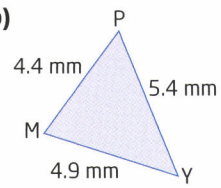
c) Compare your answers in parts a) and b). Explain why your method is more efficient.

5. Solve each triangle. Round answers to the nearest tenth of a degree.

a)



b)



6. Sketch each triangle and label it with the given information. Then, solve the triangle. Round answers to the nearest tenth of a degree.

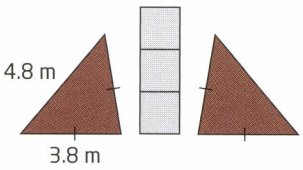
a) In acute $\triangle NBG$, $n = 15$ m, $b = 14$ m, and $g = 12$ m.

b) In acute $\triangle DRT$, $d = 5.0$ km, $r = 3.8$ km, and $t = 4.6$ km.

7. **Use Technology** Check your answers to question 6 using dynamic geometry software.

Connect and Apply

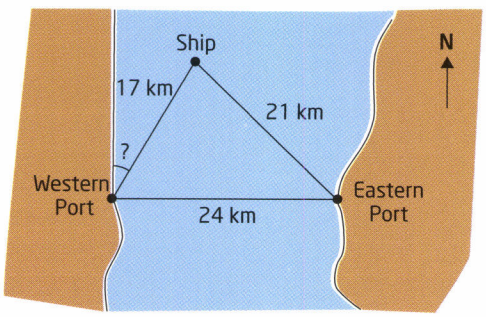
8. Ling is designing a garden for her backyard. It will consist of two congruent triangular flower beds on either side of a path, as shown.



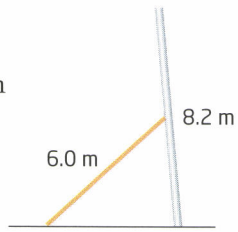
a) Find the interior angles of the flower beds, to the nearest degree.

b) Find the total area of the flower beds, to the nearest square metre.

9. A distress signal is received from a ship that is 21 km from one port and 17 km from another port. The eastern port is 24 km directly east of the western port. At what angle to the western shoreline should the ship head, in order to dock at the western port? Round to the nearest degree.



10. A leaning pole is braced at its midpoint, as shown. The pole is 8.2 m long, and the bracing beam is 6.0 m long.



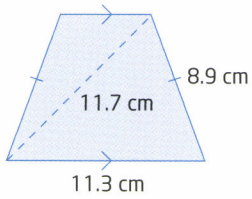
The foot of the beam is placed 5.0 m from the base of the pole.
Determine, to the nearest degree,

a) the angle the pole makes with the ground

b) the angle the beam makes with the ground

c) the angle the beam makes with the pole

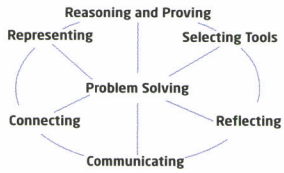
11. Find the interior angles of the isosceles trapezoid, to the nearest tenth of a degree.



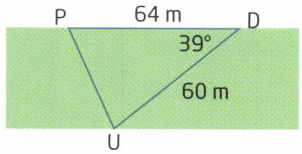
12. Create a problem involving the cosine law for which the answer is 72°.

13. Chapter Problem You and two of your team are to fly in a V-formation such that the distances between you are 80 m, 80 m, and 100 m. Find the angles that illustrate how the three aircraft should be arranged, to the nearest tenth of a degree. Is there more than one possible solution? Explain.

Achievement Check



14. Pinder, Dino, and Ursala all live on the edge of a park, as shown.



They agree to meet at one of their homes to study for a trigonometry test.

a) Is it possible to solve this triangle? Justify your response.

b) What tools will you use?

c) Whose home should they pick in order to minimize travel time? State any assumptions you make.

Extend

15. a) Acute $\triangle ABC$ is isosceles, with $b = c$. Show that $\cos A = 1 - \frac{a^2}{2b^2}$.

b) Solve $\triangle ABC$, if the equal sides are 1.5 cm and the third side is 0.8 cm.

16. Draw an equilateral triangle and label its sides a. Mark one of the angles 60°. Use the cosine law to prove that $\cos 60° = \frac{1}{2}$.

Program a Graphing Calculator

You can use graphing calculators or computer software to perform the computational steps of the sine law or the cosine law. Once you have written these programs, all you need to do is enter the given information for your triangle, and let technology do the work!

Investigate

Tools

- TI-83 Plus or TI-84 Plus graphing calculator

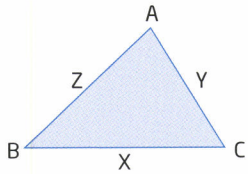
Technology Tip

Before entering and testing your programs, make sure that the graphing calculator is in Degree mode.

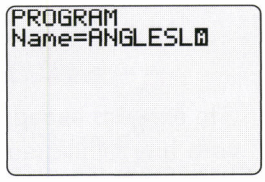
How can you program a graphing calculator to calculate an unknown measure in an acute triangle?

A: The Sine Law

You can enter a graphing calculator program that will allow you to use the sine law to find an unknown angle measure.



1. Create the program.
   - Press [PRGM].
   - Cursor over to the **NEW** menu, and select **1:Create New**.
   - Type a name that makes sense to you, such as ANGLESL.
   - Press [ENTER].



2. Enter the program.

   a) Clear the home screen.
   - Press [PRGM].
   - From the **I/O** menu, select **8:ClrHome**.
   - Press [ENTER].

   b) Ask for the known angle, ∠A, and the two known sides, X and Y.
   - Press [PRGM].
   - From the **I/O** menu, select **2:Prompt**.
   - Press [ALPHA] [A] [,] [ALPHA] [X] [,] [ALPHA] [Y].
   - Press [ENTER].

   c) Calculate the unknown angle, ∠B.
   - Press [2nd] [SIN^{-1}]. Type Y × sin A ÷ X, using the [ALPHA] key for variables and brackets as needed. Then, press [)] [STO→], and type B.
   - Press [ENTER].

d) Display the answer.

- Press PRGM.
- From the **I/O** menu, select **3:Disp**.
- Type "ANGLE B:", using ALPHA [␣] for a blank. Then, press , and type B.
- Press ENTER.

3. Test the program.

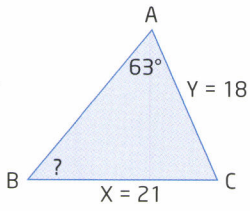
a) Press 2nd [QUIT] to return to the home screen.

b) Run the program.

- Press PRGM.
- From the **EXEC** menu, select the title of your program, for example, **ANGLESL**.
- Press ENTER twice.

You will be prompted for the known information of an acute triangle.

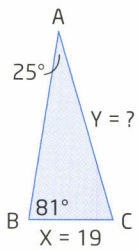
Enter the given information for △ABC and let the program calculate the measure of ∠B.



c) Verify the answer given by the program by using the sine law to calculate the value of ∠B.

4. a) Create and enter a sine law program that will allow you to find a missing side, Y, if you are given a side, X, and the two angles corresponding to X and Y, ∠A and ∠B.

b) Test the program using the triangle shown. Check the result by manual calculation.



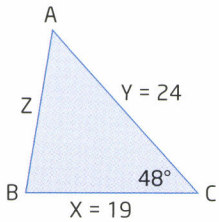
B: The Cosine Law

1. a) Create and enter the program shown.

b) What does this program do?

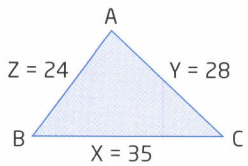
c) Test the program using △ABC. Verify the answer.

```
PROGRAM:SIDECL
:ClrHome
:Prompt X,Y,C
:√(X²+Y²-2XYcos(
C))→Z
:Disp "SIDE Z:",
Z
:
```



2. a) Create and enter a program that will allow you to find an unknown angle of an acute triangle if you are given three sides.

b) Test the program using the triangle shown. Verify the answer.



Program a TI-89 Calculator

The following instructions show how to program a TI-89 calculator. The same logic can be used for other programmable calculators or computer software.

Investigate

Tools

- TI-89 calculator

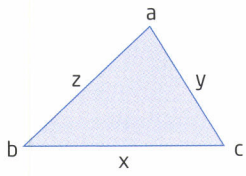
Technology Tip

Before entering and testing your programs, make sure that the calculator is set to DEGREE for Angle mode and APPROXIMATE for Exact/Approx mode.

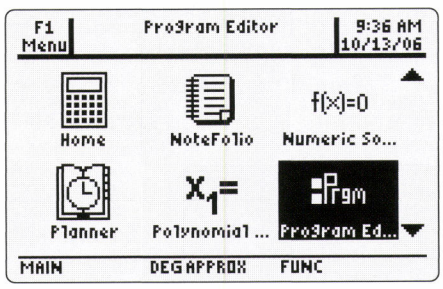
How can you program a TI-89 calculator to calculate an unknown measure in an acute triangle?

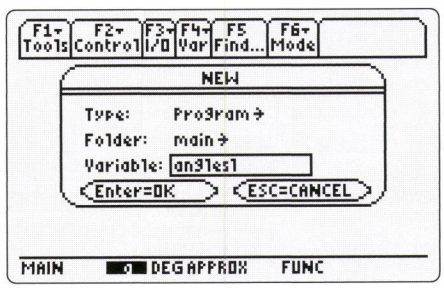
A: The Sine Law

You can enter the following program to use the sine law to find an unknown angle measure.



1. Create the program.
   - Press APPS, and select **Program Editor**.
   - Select **3:New...**.
   - Cursor down to the **Variable:** box. Type a name that makes sense to you, such as anglesl.
   - Press ENTER twice.





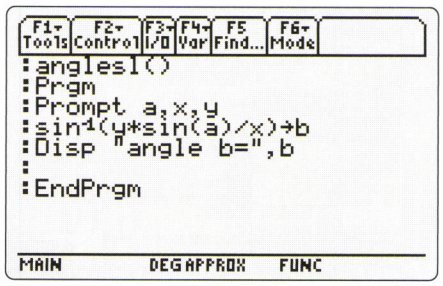
2. Enter the program.
   a) Ask for the known angle, ∠a, and the two known sides, x and y.
   - Cursor down to the blank line after Prgm.
   - Press F3 and select **5:Prompt**.
   - Press ALPHA [A] , X , Y .
   - Press ENTER.

b) Calculate the unknown angle, ∠b.

- Press ◆ [SIN^{-1}]. Type y × sin a ÷ x, using the ALPHA key and brackets as needed. Then, press) STO▸, and type b.
- Press ENTER.

c) Display the answer.

- Press F3 and select **2:Disp**.
- Type "angle b=", using 2nd 1 for ["] and ALPHA [␣] for a blank. Then, press , and type b.
- Press ENTER.



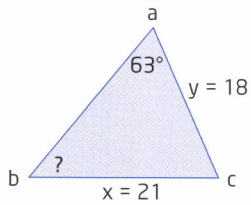
3. Test the program.

a) Press HOME to return to the Home screen.

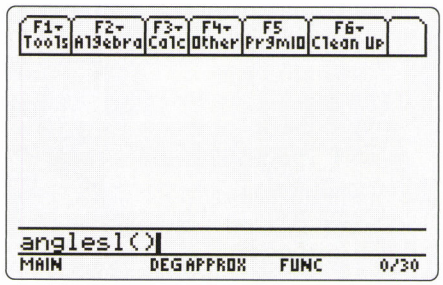
b) Run the program.

- Type anglesl().
- Press ENTER.

You will be prompted for the known information of an acute triangle.



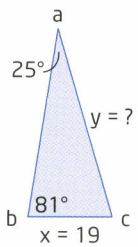
Enter the given information for the triangle shown and let the program calculate the measure of ∠b.



c) Verify the answer given by the program by using the sine law to calculate the value of ∠b.

4. a) Create and enter a sine law program that will allow you to find a missing side, y, if you are given a side, x, and the two angles corresponding to x and y, ∠a and ∠b.

b) Test the program using the triangle shown. Check the result by manual calculation.

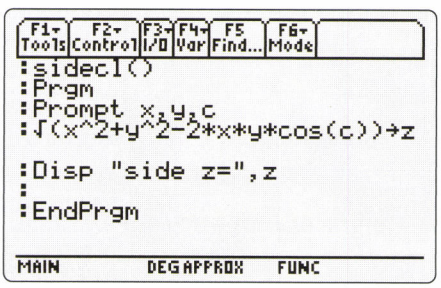


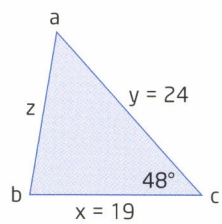
B: The Cosine Law

1. a) Create and enter the program shown.

b) What does this program do?

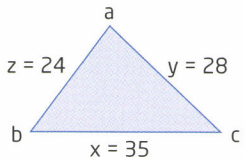
c) Test the program using the triangle shown. Verify the answer.





2. a) Create and enter a program that will allow you to find an unknown angle of an acute triangle if you are given three sides.

b) Test the program using the triangle shown. Verify the answer.

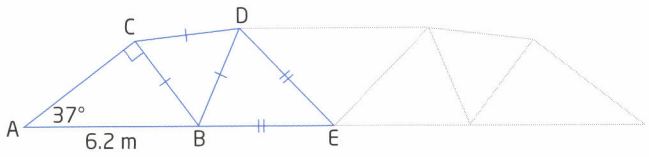


8.4 Solve Problems Using Trigonometry

Engineers, scientists, and architects apply a variety of mathematical tools, including trigonometry. When solving problems or creating designs, they must be able to efficiently combine algebraic and geometric reasoning. It is also important for them to be able to communicate their ideas to others effectively.

Example 1 Bridge Truss

A section of a bridge truss design is shown. Find the total length of the beams required to build the section, to the nearest tenth of a metre.



Solution

The section of the truss consists of three triangles. Solve for each side length using trigonometry and other mathematical tools.

Start with the triangle with the most given information, $\triangle ABC$.

Notice that $\triangle ABC$ is a right triangle. First, use the sine ratio to find a. Then, use the cosine ratio to find b.

$$\frac{a}{c} = \sin A \qquad \frac{b}{c} = \cos A$$

$$\frac{a}{6.2} = \sin 37° \qquad \frac{b}{6.2} = \cos 37°$$

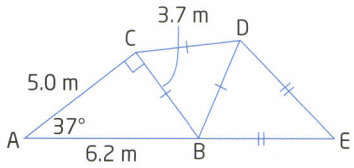
$$a = 6.2(\sin 37°) \qquad b = 6.2(\cos 37°)$$

$$a \doteq 3.7 \qquad b \doteq 5.0$$

I can use the Pythagorean theorem to check.

$c^2 = 3.7^2 + 5.0^2$
$c^2 = 38.44$
$c \doteq 6.2$

Label these lengths on the diagram.



Focus on △BCD:

Notice that △BCD is an equilateral triangle. All sides have equal length. Therefore,
BC = CD = DB = 3.7 m

Focus on △BDE:

Notice that △BDE is an isosceles triangle. You can apply the sine law, but you need to find ∠EBD first. The three angles at point B are supplementary.

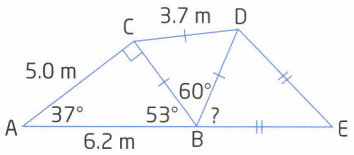
In △ABC, ∠CBA and ∠BAC are complementary because the third angle is 90°.

$$\begin{aligned}\angle CBA &= 90° - 37° \\ &= 53°\end{aligned}$$

Since △BCD is an equilateral triangle, the three interior angles are equal. Therefore,
∠CBD = 60°

Use the two known angles at B to find ∠DBE.

$$\begin{aligned}53° + 60° + \angle DBE &= 180° \\ \angle DBE &= 180° - 53° - 60° \\ \angle DBE &= 67°\end{aligned}$$



△BDE is isosceles with BE = DE. So, ∠EDB = ∠DBE = 67°.

Use this information to find ∠E.

$$\begin{aligned}\angle E &= 180° - 2(67°) \\ &= 46°\end{aligned}$$

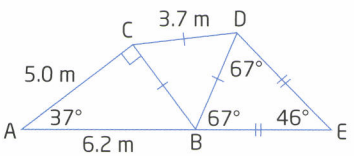
Now use the sine law in △BDE to find the lengths of beams DE and BE.

$$\frac{b}{\sin B} = \frac{e}{\sin E}$$

$$\frac{b}{\sin 67°} = \frac{3.7}{\sin 46°}$$

$$b = \sin 67° \left(\frac{3.7}{\sin 46°}\right)$$

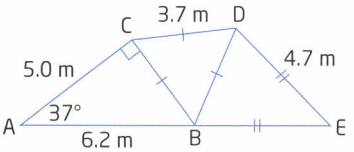
$$b \doteq 4.7$$



To find the total length of the beams required to build this section of the truss, add all the lengths.

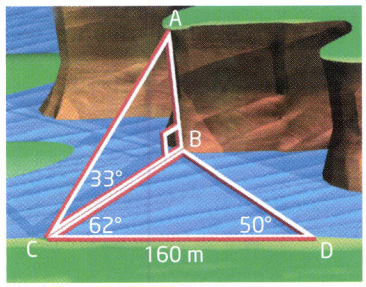
$$\begin{aligned}\text{Total length} &= 5.0 + 6.2 + 3(3.7) + 2(4.7) \\ &= 31.7\end{aligned}$$

The total length of the beams required is 31.7 m.



Example 2 Height of a Cliff

Find the height of the cliff shown, to the nearest metre.



Literacy Connections

In complicated diagrams involving more than one triangle, using one letter to identify side lengths can be confusing. In this example, side *b* could mean side AC or side CD. In such cases, use the two endpoints of a line segment to identify it. Similarly, use three letters to identify angles, as needed.

Solution

There is not enough given information in $\triangle ABC$ to solve for the height directly. Use the given information in $\triangle BCD$ to solve for the width of the river, BC. Then, use this to find the height of the cliff.

Focus on $\triangle BCD$:

Find the measure of $\angle CBD$.

$$\begin{aligned}\angle CBD &= 180° - 62° - 50° \\ &= 68°\end{aligned}$$

Now, use the sine law to find the width of the river, BC.

$$\frac{BC}{\sin BDC} = \frac{CD}{\sin CBD}$$

$$\frac{BC}{\sin 50°} = \frac{160}{\sin 68°}$$

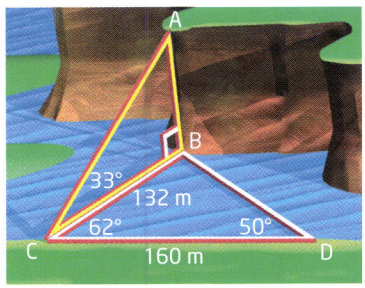
$$BC = \frac{160}{\sin 68°}(\sin 50°)$$

$$BC \doteq 132$$

The width of the river is about 132 m. Use this to find the height of the cliff, AB.

Focus on $\triangle ABC$:

When working with right triangles, the sine law and cosine law still apply. However, it is easier to apply the primary trigonometric ratios in right triangles.



$$\tan ACB = \frac{AB}{BC}$$

$$\tan 33° = \frac{AB}{132}$$

$$132(\tan 33°) = AB$$

$$AB \doteq 86$$

The height of the cliff is approximately 86 m.

Key Concepts

- When solving problems involving right triangles, you can apply the primary trigonometric ratios.
- When solving problems involving acute triangles, you can apply the sine law or the cosine law:
  - Use the sine law if you are given an angle and the opposite side, plus one other side or angle.
  - Use the cosine law if you are given two sides and the contained angle, or three sides.
- A number of problems involving trigonometry require multiple steps. Look for techniques and make connections to other branches of mathematics, such as geometry, in order to solve problems efficiently.

Communicate Your Understanding

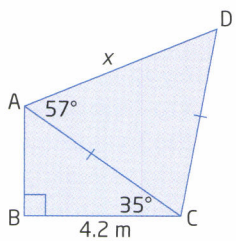
C1 **a)** Do the sine law and the cosine law hold true in right triangles? Explain.

b) What other techniques can you use to solve right triangles?

C2 Explain how you can decide whether to apply the sine law or the cosine law in an acute triangle.

C3 **a)** Describe the steps you would take to find the length of x in the diagram shown.

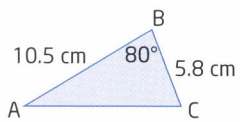
b) Describe a different set of steps that will also work.



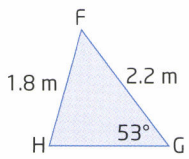
Practise

1. Determine whether the primary trigonometric ratios, the sine law, or the cosine law should be used first to solve each triangle.

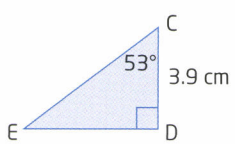
a)



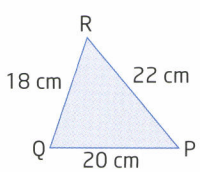
b)



c)



d)



For help with questions 2 and 3, see Examples 1 and 2.

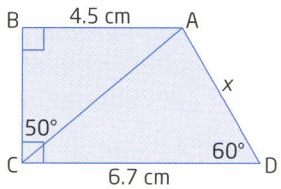
2. Refer to question C3.

a) Use your method from part a) to find x, to the nearest tenth of a centimetre.

b) Find x using another method. Compare your answers. Are they equal?

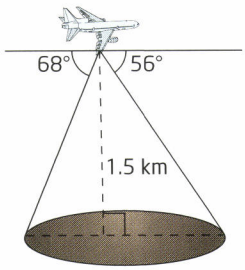
3. **a)** Find x, to the nearest tenth of a centimetre.

b) Find x using a different method.



Connect and Apply

4. While flying at an altitude of 1.5 km, a plane measures angles of depression to opposite ends of a large crater, as shown. Find the width of the crater, to the nearest tenth of a kilometre.



5. Earth is 149 600 000 km from the Sun. This distance is equal to 1 A.U. (astronomical unit). Mars is 1.5 A.U. from the Sun. One evening, Mars is seen from Earth to make an angle of 68° with the Sun.

a) Draw a diagram and label the given information.

b) How far apart are Earth and Mars at this point, in kilometres?

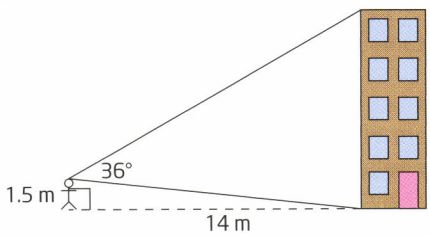
c) Do you think the distance between Earth and Mars is always the same? Explain why or why not.

6. Lena is in a bicycle road race. In the first leg, she rides 12 km from Riverside to Danton. Then, she turns and rides 17 km to Humberville, making a 74° angle from the first leg. The final turn leads back to Riverside.

a) What is the total length of the race, to the nearest kilometre?

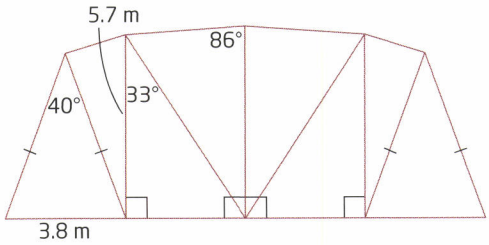
b) At what angles are the three towns situated with respect to each other? Round to the nearest degree.

7. Trevor, who is 1.5 m tall, is standing at a distance of 14 m from a building. From his point of view, the bottom and top of the building are separated by 36°, as shown. How tall is the building, to the nearest tenth of a metre?



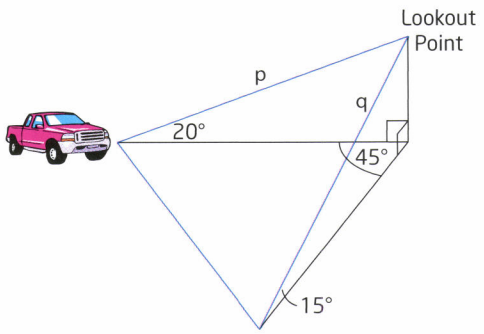
8. Rocco and Biff are two koala bears frolicking in a meadow. Suddenly, a tasty clump of eucalyptus falls to the ground, catching their attention. Biff glances at Rocco, who appears to be 15 m away, then over to the eucalyptus, which appears to be 18 m away. From Biff's point of view, Rocco and the eucalyptus are separated by an angle of 45°. Rocco's top running speed is 1.0 m/s, but Biff can run one and a half times as fast. Can Biff beat Rocco to the eucalyptus? State any assumptions you make.

9. Find the total length of materials required to build the bridge truss shown, to the nearest tenth of a metre.

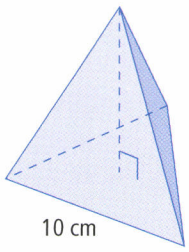


Describe the steps in your solution and state any assumptions you make.

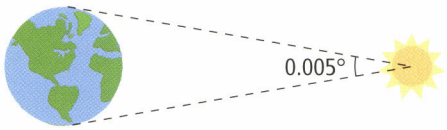
10. Lookout Point is accessible from two trails, both of which start from the same altitude and climb upward. Path p travels east to the point and climbs at an average angle of elevation of 20°. Path q travels northeast to the point at an average angle of elevation of 15°. Path p is 2.0 km long. Jack and Debbie parked at the base of path p. They hiked a round trip up path p to Lookout Point, then down path q, and then finally straight from the base of path q back to their truck. How far did they hike, to the nearest tenth of a kilometre? State any assumptions you make.



11. A tetrahedron has edges that are 10 cm in length. Find the height of this tetrahedron, to the nearest tenth of a centimetre.



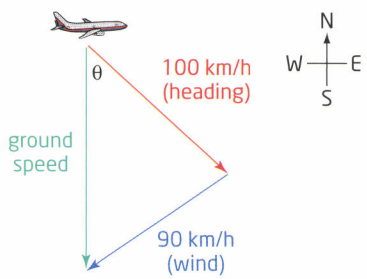
12. Doctors Jones and Hwang are astronomers observing the sun from opposite ends of Earth. The radius of Earth is 6400 km.



a) Use this information to verify the distance from Earth to the Sun, which was given in question 5. State any assumptions you make.

b) At approximately what times of day were these observations made by each astronomer? Explain your answer.

13. **Chapter Problem** Pilots must take wind into account when flying, or the wind will blow them off course and they will not reach the desired destination. Your aircraft cruises at a speed of 100 km/h. There is a strong wind blowing from N60°E at a speed of 90 km/h. You need to fly south to home base.

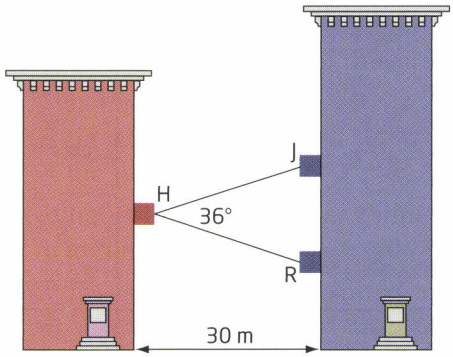


a) Find the direction, θ, you must aim the plane, to the nearest degree.

b) What will your speed be, over the ground? Round to the nearest unit.

Extend

14. Helen, Javier, and Raquel live in two identical apartment buildings, located 30 m apart. Javier lives two floors higher than Helen. Raquel lives four floors lower than Helen. There is a 36° angle of separation when Helen looks from her balcony to those of her two friends.



a) How far apart, vertically, do Javier and Raquel live? Round to the nearest tenth of a metre.

b) Explain how you solved this problem and discuss any assumptions you made.

15. A box is in the shape of a square-based prism. The height of the box is twice the width of the base.

a) Show that the longest thin rod that can be encased in the box has length $\sqrt{6}w$, where w is the width of the base.

b) Find the angles that such a rod would make with each edge of the box.

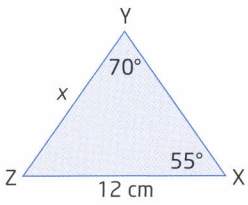
16. A ship travels 100 km at a bearing of N60°E and then turns and travels 80 km at a bearing of S20°E before reaching its destination. Suppose the ship travelled directly from its starting point to its destination, following a direct route. What distance and at what bearing would the ship travel? Round to the nearest unit.

17. Who uses trigonometry in their careers? Do some research to find out what types of careers require the use of trigonometry and why. Write a brief report of your findings.

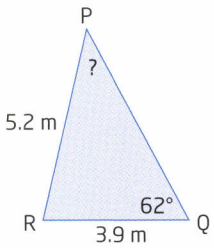
Chapter 8 Review

8.1 The Sine Law, pages 396–404

1. Find the length of x, to the nearest centimetre.

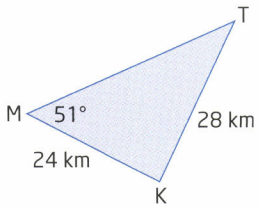


2. Find the measure of $\angle P$, to the nearest degree.



3. Solve each triangle. Round answers to the nearest unit, if necessary.

a)



b) In acute $\triangle NRC$, $\angle N = 47°$, $\angle R = 80°$, and $r = 27$ mm.

4. Three billiard balls are lying as shown.

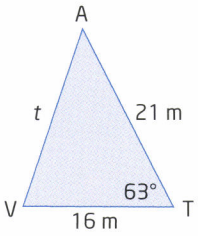
The white cue ball is 55 cm from the red ball and 62 cm from the black ball. The line segment from the red ball to the white ball and the line segment from the red ball to the black ball form an angle of 82°.

a) How far apart are the red and black balls, to the nearest centimetre?

b) Find the measures of the other angles in the triangle formed by these three billiard balls, to the nearest degree.

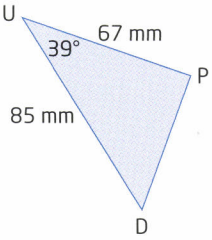
8.2 The Cosine Law, pages 405–411

5. Find the length of t, to the nearest metre.



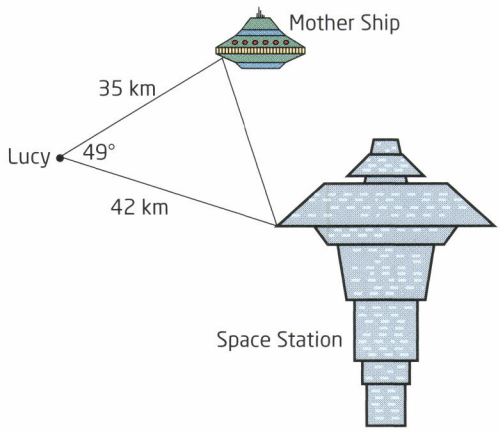
6. Solve each triangle. Round answers to the nearest unit, if necessary.

a)



b) In acute $\triangle EQW$, $\angle W = 77°$, $e = 11$ km, and $q = 14$ km.

7. Lucy's space shuttle is 35 km from her mother ship and 42 km from a space station. From Lucy's point of view, the mother ship and space station appear to be 49° apart, as shown.

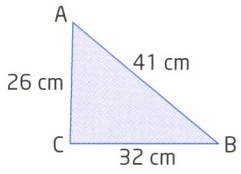


a) How far apart are the mother ship and the space station, to the nearest kilometre?

b) From the space station, by what angle do the shuttle and mother ship appear to be separated, to the nearest degree?

8.3 Find Angles Using the Cosine Law, pages 412–419

8. Find the measure of $\angle B$, to the nearest degree.

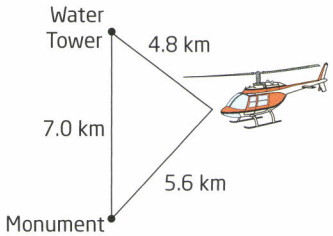


9. Solve each triangle. Round answers to the nearest tenth of a degree.

a) In acute $\triangle VSF$, $v = 2.9$ km, $s = 3.5$ km, and $f = 3.0$ km.

b) In acute $\triangle SBZ$, $s = 19$ m, $b = 21$ m, and $z = 13$ m.

10. While flying, a helicopter pilot spots a water tower that is 4.8 km away to the north. At the same time, she also sees a monument that is 5.6 km away to the south. The tower and the monument are separated by a distance of 7.0 km along the flat ground. Find the angles at which the pilot is viewing the water tower and the monument, to the nearest degree.



8.4 Solve Problems Using Trigonometry, pages 424–429

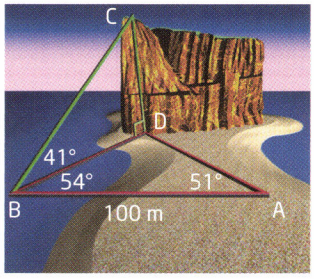
11. A person on the ground is directly between two helicopters that are flying toward each other at the same altitude. The first helicopter is 2.0 km away from the observer, at an angle of elevation of 30°, while the second helicopter is 3.5 km away. Round answers to the nearest tenth of a kilometre, if necessary.

a) Draw a diagram and label the known information.

b) How far apart are the helicopters?

c) What is their altitude?

12. Percé Rock is located on the shore of the Gaspé Peninsula in Québec. The measurements shown were taken at low tide.



Find the height of Percé Rock, to the nearest metre.

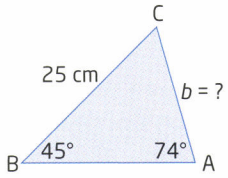
Chapter Problem Wrap-Up

Congratulations! You have passed your first course at Fowler's Aeronautical School. Your last assignment is to create an aeronautical problem of your own involving trigonometry for next year's class. Once you have created the problem, solve it. Then, trade with a partner and solve each other's problem.

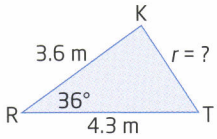
WWW If you are interested in finding the next steps you need to take in order to become a pilot, go to www.mcgrawhill.ca/links/principles10 and follow the links.

Chapter 8 Practice Test

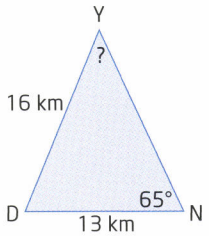
1. Find the length of the indicated side, to the nearest centimetre.



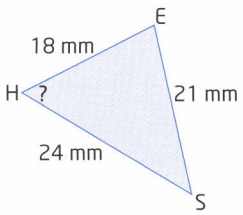
2. Find the length of the indicated side, to the nearest tenth of a metre.



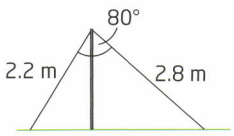
3. Find the measure of the indicated angle, to the nearest degree.



4. Find the measure of the indicated angle, to the nearest degree.



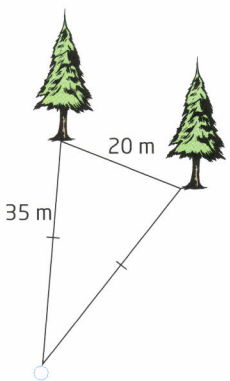
5. Two wires are supporting a tent pole, as shown.



a) How far apart are the wires fixed in the ground, to the nearest tenth of a metre?

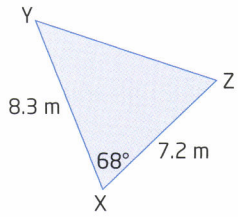
b) Find the angle each wire makes with the ground, to the nearest degree. State any assumptions you make.

6. Cheryl is trying to hit her golf ball between two trees. She estimates the distances shown.



Within what angle must Cheryl make her shot, in order to pass between the trees? Round to the nearest tenth of a degree.

7. Solve $\triangle XYZ$. Round the side length to the nearest tenth of a metre and the angle measures to the nearest degree.

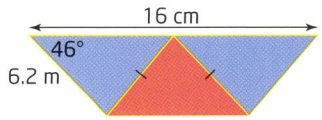


8. In acute $\triangle NTW$, $\angle N = 54°$, $n = 2.3$ km, and $w = 1.8$ km.

a) Sketch this triangle and label the given information.

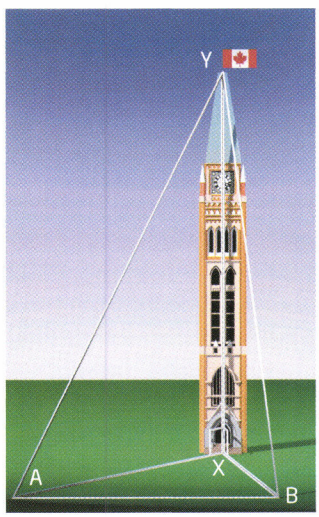
b) Solve this triangle. Round the side length to the nearest tenth of a metre and the angle measures to the nearest degree.

9. Students of Fowler's Aeronautical School have this crest on their pilot's cap. Find the total length of gold trim needed to make a crest, to the nearest tenth of a centimetre. State any assumptions you make.



10. Use the measurements given to find the height of the Peace Tower in Ottawa, to the nearest metre.

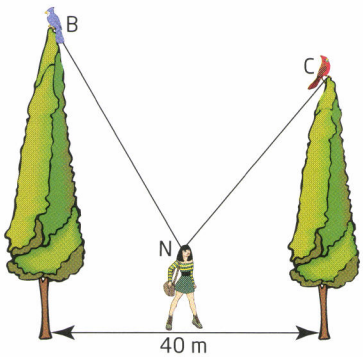
$AB = 50$ m
$\angle XAY = 43°$
$\angle XAB = 60°$
$\angle ABX = 82°$



11. Tess was flying from Toronto to Hamilton at night. She noticed that her heading indicator was malfunctioning and decided to check her position. She called Hamilton Tower and St. Catharines Radio, knowing that the stations were 50 km apart. Hamilton reported that the position of her plane formed an angle of 65° with the line joining the two stations. St. Catharines reported that her position made an angle of 48°. How far was the plane from Hamilton, to the nearest kilometre?

12. Scuba divers count fin strokes to estimate the distance they have travelled under water. Felipe uses this method and a compass to swim 300 m north from the dive boat. He then turns right 120° and swims 400 m. How far from the boat is he, to the nearest metre? In what direction must Felipe swim to return to the boat?

13. While standing exactly halfway between two tall trees, Nikki spots a blue jay at the top of one tree and a cardinal at the top of the other, as shown.

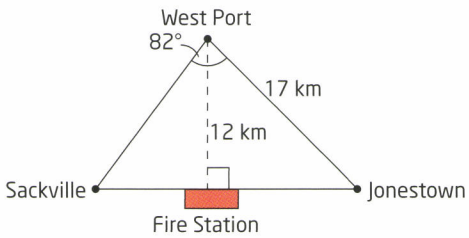


From Nikki's point of view, the angles of elevation to the blue jay and to the cardinal are 60° and 50°, respectively. The trees are 40 m apart.

a) If Nikki is 1.5 m tall, find the height of each tree, to the nearest tenth of a metre.

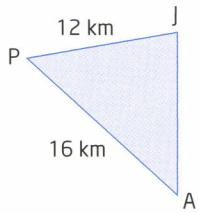
b) How far apart are the two birds, to the nearest tenth of a metre?

14. A fire station serves an area bordered by three highways that join the towns of West Port, Sackville, and Jonestown, as shown.



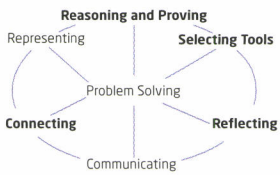
The fire station serves a total area of 126 km^2. Find the number of minutes it would take for a fire truck to drive the perimeter of this region, assuming an average speed of 80 km/h.

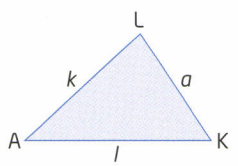
15. From his cockpit, the pilot of a flying plane observes a jet flying directly above an airport, as shown. From the plane, the angle of elevation to the jet is 14° and the angle of depression to the airport is 38°. Find the altitude of both aircraft, to the nearest kilometre.



Achievement Check

16. A small lake has three approximately straight shorelines so it forms a triangle, △LAK.





a) List all of the possible measurements that could be determined for this lake.

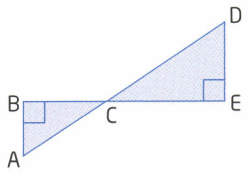
b) If you were a surveyor standing at point L, which of these measurements would you measure and which ones would you calculate? Justify your choice.

c) Could you have made a different choice in part b)? Is this choice better than your previous one? Explain.

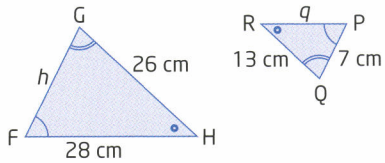
Chapters 7 and 8 Review

Chapter 7 Trigonometry of Right Triangles

1. Name the two similar triangles and explain why they are similar.

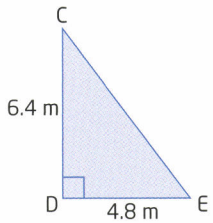


2. The two triangles are similar. Find the unknown side lengths.

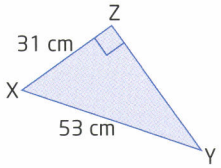


3. Find the measure of both acute angles in each triangle, to the nearest degree.

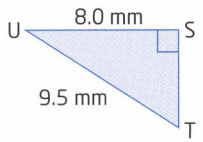
a)



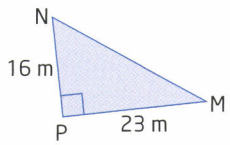
b)



c)

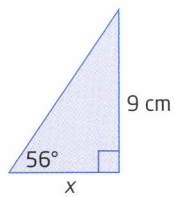


d)

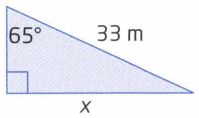


4. Find the length of x, to the nearest tenth of a unit.

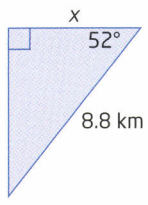
a)



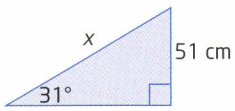
b)



c)

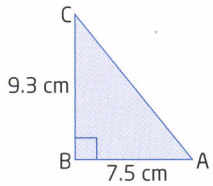


d)

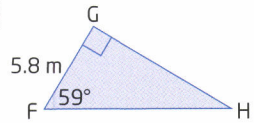


5. Solve each triangle. Round each side length to the nearest tenth of a unit and each angle to the nearest degree.

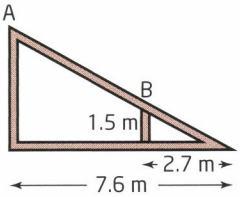
a)



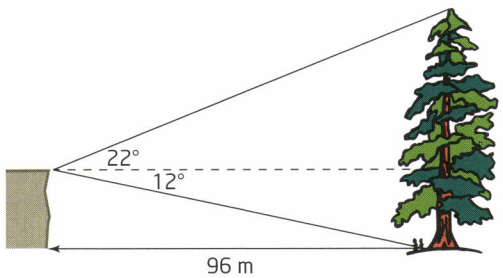
b)



6. In a roof, a 1.5-m support is to be placed at point B, as shown. Find the length of the support to be placed at point A, to the nearest tenth of a metre.



7. From a rock ledge, the angle of elevation to the top of a tree is 22°. The angle of depression to the base of the tree is 12°.

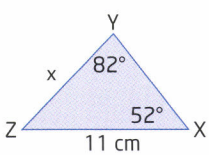


a) Find the height of the rock ledge, to the nearest metre.

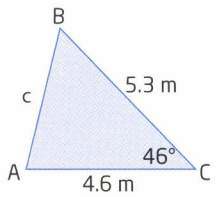
b) Find the height of the tree, to the nearest metre.

Chapter 8 Trigonometry of Acute Triangles

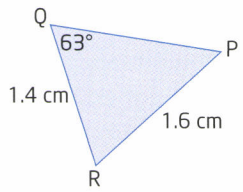
8. Find the length of x, to the nearest centimetre.



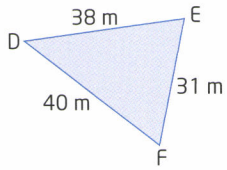
9. Find the length of c, to the nearest tenth of a metre.



10. Find the measure of $\angle P$, to the nearest degree.

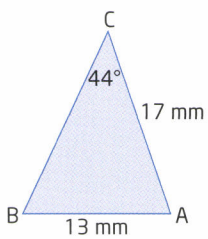


11. Find the measure of $\angle E$, to the nearest degree.

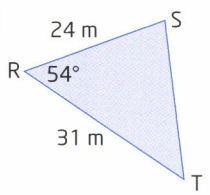


12. Solve each triangle. Round answers to the nearest unit, if necessary.

a)



b)



c) In acute $\triangle WXY$, $\angle X = 81°$, $\angle W = 32°$, and $w = 16$ cm.

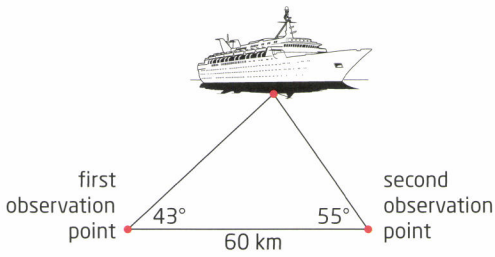
d) In acute $\triangle EFG$, $\angle E = 84°$, $f = 32$ km, and $g = 21$ km.

13. Solve each triangle. Round answers to the nearest tenth of a degree.

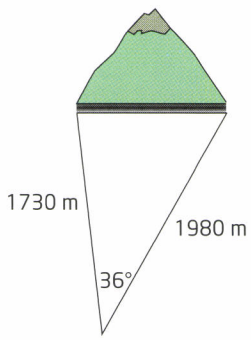
a) In acute $\triangle ABC$, $a = 6.8$ cm, $b = 8.7$ cm, and $c = 9.6$ cm.

b) In acute $\triangle TUV$, $t = 10.3$ m, $u = 11.4$ m, and $v = 12.5$ m.

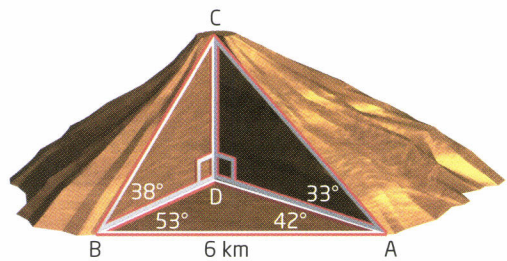
14. A ship is sighted at sea from two observation points on the coastline that are 60 km apart. The angle between the coastline and the ship at the first observation point is 43°. From the second observation point, the angle between the coastline and the ship is 55°. How far is the ship from the second observation point, to the nearest kilometre?



15. Find the length of the tunnel, to the nearest metre.



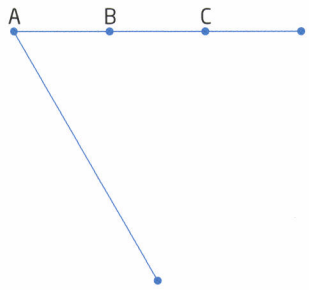
16. Find the height of the mountain, to the nearest metre.



Tasks

Solar Lights

Solar-powered lighting is an attractive and popular way of providing illumination for paths and walkways. In the Queen's Gardens, two pathways meet at an angle of 60°, as shown.



Solar lights are placed 5 m apart on one of the paths at points A, B, and C. These lights have an illumination radius of 7 m.

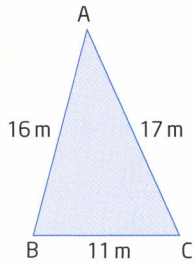
a) What length of the second path will be illuminated by light A, to the nearest metre?

b) What length of the second path will be illuminated by lights A and B, to the nearest metre?

c) What length of the second path will be illuminated by all three lights, to the nearest metre?

Lighting the Park

Julia works for the city parks department. She has been given the task of placing solar lights to illuminate the paths around the edges of a featured triangular flower bed.



What is the least number of solar lights with an illumination radius of 7 m needed for the task? Where should Julia place them, to the nearest tenth of a metre? Justify your conclusions.

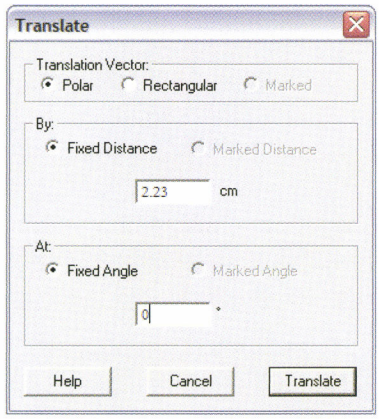
Trigonometry Using *The Geometer's Sketchpad*®

All trigonometry problems are based on solving triangles, given certain information. Three common situations involve solving a triangle given measures for side-angle-side, side-side-side, and angle-side-angle. These problems could be solved using the sine and cosine laws. *The Geometer's Sketchpad*® provides an alternative method of solving trigonometry problems with a high degree of precision.

The precision of measurements in *The Geometer's Sketchpad*® can be set to the nearest hundred thousandth of a unit, which is more accurate than most commonly available measuring devices. In addition, sides and angles can be drawn precisely using the **Transform** menu.

For example, to draw $\triangle ABC$ with $\angle A = 23.5°$, AB = 2.23 cm, and AC = 3.0 cm:

- Open a new sketch. Construct point A. Select point A. From the **Transform** menu, choose **Mark Center**.
- Select point A. From the **Transform** menu, choose **Translate...**. In the Translate dialogue box, choose **Polar**, and enter Fixed Distance 2.23 and Fixed Angle 0.0. Click on **Translate**. Label the new point B.
- Again, select point A. From the **Transform** menu, choose **Translate...**. In the Translate dialogue box, choose **Polar**, and enter Fixed Distance 3.0 and Fixed Angle 23.5. Click on **Translate**. Label the new point C.
- Join points A, B, and C.



Then, to solve $\triangle ABC$, use the **Measure** menu to find the measures of $\angle B$, $\angle C$, and side BC.

a) Use *The Geometer's Sketchpad*® to solve acute $\triangle DEF$, where $\angle E = 55.2°$, $\angle F = 63.0°$, and EF = 3.2 cm.

b) Use *The Geometer's Sketchpad*® to solve acute $\triangle XYZ$, where $\angle X = 40.5°$, XY = 1.45 cm, and XZ = 1.74 cm.

c) Use *The Geometer's Sketchpad*® to solve acute $\triangle GHI$, where GH = 2.567 cm, HI = 2.869 cm, and GI = 3.473 cm.

d) Comment on the advantages and disadvantages of using *The Geometer's Sketchpad*® to solve these problems, compared to using the sine and cosine laws.

Course Review

Chapter 1 Linear Systems

1. Translate each sentence into an equation. Tell how you are assigning the two variables.

a) The perimeter of a basketball court is 40 m.

b) The average of two numbers is 15.

c) The value of the quarters and loonies in a vending machine is $37.

d) The total receipts from adult tickets at $20 each and student tickets at $12 each was $9250.

2. Use graphing to find the point of intersection of each pair of lines.

a) $x - y = 4$
$3x + 2y = 7$

b) $y = \frac{1}{2}x - 4$
$y = 2x - 1$

c) $x + y - 4 = 0$
$5x - y - 8 = 0$

3. Solve each linear system using the method of substitution.

a) $x + 4y = 6$
$2x - 3y = 1$

b) $y = 6 - 3x$
$y = 2x + 1$

c) $5x - y = 4$
$3x + y = 4$

4. Solve by elimination. Check each solution.

a) $x + y = 55$
$2x - y = -4$

b) $2a + b = 5$
$a - 2b = 10$

c) $4k + 3h = 12$
$4k - h = 4$

d) $5a - 2b = 5$
$3a + 2b = 19$

5. Explain why the following linear system has no solution.

$y - 2x = 1$
$y = 2x + 3$

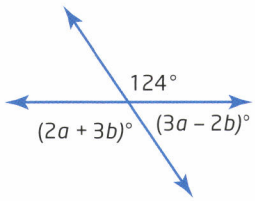
6. Use a graphing calculator to find the point of intersection of each pair of lines. Round your answers to the nearest tenth.

a) $y = x - 5$
$x + 2y = 10$

b) $2x + 5y + 20 = 0$
$5x - 3y + 15 = 0$

c) $y = 7x$
$3y = 5x - 2$

7. Find the values of a and b in the diagram shown.



8. A boat took 5 h to travel 60 km up a river, against the current. The return trip took 3 h. Find the speed of the boat in still water and the speed of the current.

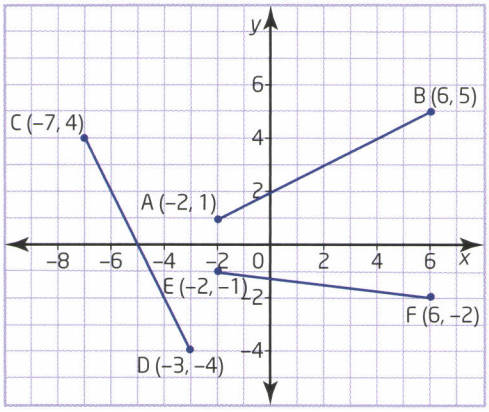
9. What volumes of 60% hydrochloric acid solution and 30% hydrochloric acid solution must be mixed to make 125 mL of 36% hydrochloric acid solution?

10. Solve the linear system.

$$\frac{x-2}{3} + \frac{y+1}{5} = 2$$
$$\frac{x+2}{7} - \frac{y+5}{3} = -2$$

Chapter 2 Analytic Geometry

11. Find the coordinates of the midpoint and the length of each line segment.

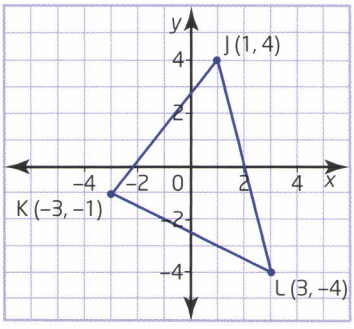


12. For △JKL, find an equation for

a) the median from vertex J

b) the median from vertex K

c) the right bisector of side JL



13. On a street map, the coordinates of the two fire stations in a town are A(10, 63) and B(87, 30). A neighbour reports smoke coming from the kitchen of a house at C(41, 18).

a) Which fire station is closer to this house?

b) Describe how to use geometry software to answer part a).

14. Use analytic geometry to classify the quadrilateral with vertices D(10, 0), E(2, 4), F(−8, −6), and G(6, −8). Explain your reasoning and show all your work.

15. a) Draw the triangle with vertices J(2, 10), K(6, −6), and L(14, 6).

b) Calculate the coordinates of the midpoint, M, of side JK and the coordinates of the midpoint, N, of side JL.

c) Show that MN is half the length of KL.

d) Show that MN is parallel to KL.

16. Does the point P(−3, −2) lie on the right bisector of the line segment with endpoints Q(−2, 5) and R(4, 1)? Justify your answer.

17. a) Determine the type of quadrilateral that has vertices at A(4, 6), B(−4, −2), C(2, −5), and D(11, 4).

b) Describe how to use geometry software to verify your answer to part a).

18. On a site plan for a new house, a water main runs along the edge of the property straight from point W(10, 34) to point M(2, 2). The water service will enter the house at point H(24, 22). The grid intervals on the plan represent 0.5 m.

a) At what point on the water main should the connector to take water to the house be located? Explain your reasoning.

b) What length of pipe will this connection require?

19. Find an equation for the circle centred at the origin and passing through the point

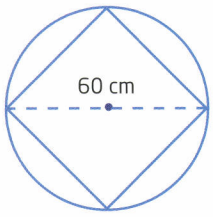
a) J(0, 7)

b) K(5, 6)

c) $(8, \sqrt{3})$

20. Find the diameter and the area of the circle defined by $x^2 + y^2 = 64$.

21. You are designing a chute for loading grain into rail cars. The cars have a round hatch 60 cm in diameter. The chute will have a square cross section. Find the side length of the largest chute that will fit into the hatch. Round your answer to the nearest centimetre.



Chapter 3 Geometric Properties

22. a) What is the centroid of a triangle?

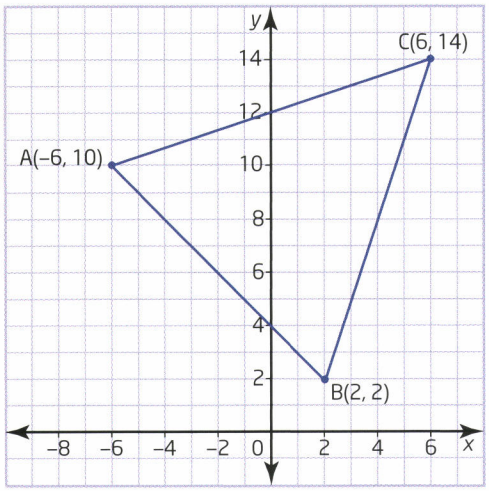
b) Describe how to use analytic geometry to find the coordinates of the centroid of a triangle, given the coordinates of the vertices.

c) Outline how to use geometry software to find the centroid of a triangle.

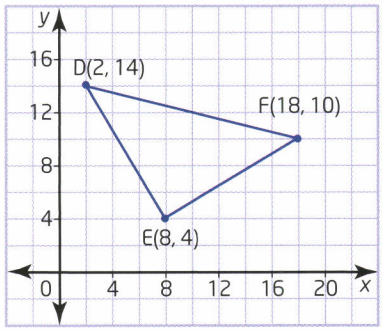
23. a) Show that each median of a triangle bisects the area of the triangle.

b) Outline how to use geometry software to answer part a).

24. Verify that $\triangle ABC$ is isosceles.



25. a) Verify that $\triangle DEF$ is a right triangle.

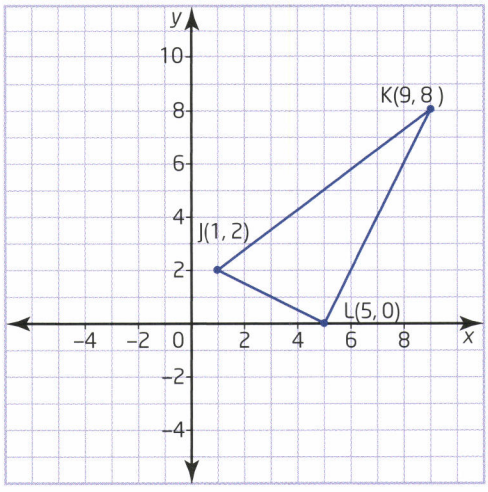


b) Describe another way to use analytic geometry to answer part a).

26. a) Draw the triangle with vertices J(−4, −6), K(2, −4), and L(8, 10).

b) Verify that the triangle formed by joining the midpoints of the sides of $\triangle JKL$ is similar to $\triangle JKL$.

27. a) Find an equation for each of the right bisectors of the sides of $\triangle JKL$.



b) Find the circumcentre of $\triangle JKL$, the point of intersection of the right bisectors of the three sides.

c) Show that the circumcentre of $\triangle JKL$ is equidistant from its vertices.

28. List the types of quadrilaterals that have diagonals that

a) are equal in length

b) bisect each other

c) meet at right angles

29. a) Show that a line segment that joins the midpoints of opposite sides of a parallelogram has the same length as the other two sides of the parallelogram.

b) Outline how to use geometry software to answer part a).

30. a) Verify that the quadrilateral with vertices A(2, 7), B(−2, 2), C(4, −2), and D(8, 3) is a parallelogram.

b) Verify that the diagonals of ABCD are equal in length.

c) Verify that the diagonals bisect each other.

d) Verify that the diagonals are not perpendicular.

31. Determine the type of quadrilateral created by the points of intersection of the lines $x - y + 3 = 0$, $y = x - 2$, $y = -\frac{1}{2}x + 5$, and $x + 2y + 12 = 0$.

32. a) Verify that the points P(2, 5), Q(6, −1), and R(7, 4) all lie on a circle centred at C(4, 2).

b) Verify that the centre of the circle lies on the right bisector of chord PQ.

33. Outline how to use geometry software to find the centre of a circle given the coordinates of three points on the circle.

Chapter 4 Quadratic Relations

34. Use finite differences to determine whether each relation is linear, quadratic, or neither.

a)

x	y
−2	−12
−1	−3
0	0
1	−3
2	−12
3	−36

b)

x	y
−2	−6
−1	−3
0	0
1	3
2	6
3	9

c)

x	y
−2	−12
−1	−7
0	−2
1	3
2	8
3	13

35. A basketball shot is taken from a horizontal distance of 5 m from the hoop. The height of the ball can be modelled by the relation $h = -7.3t^2 + 8.25t + 2.1$, where h is the height, in metres, and t is the time, in seconds, since the ball was released.

a) From what height was the ball released?

b) What was the maximum height reached by the ball?

c) If the ball reached the hoop in 1 s, what was the height of the hoop?

36. Sketch each parabola. Label the coordinates of the vertex and the equation of the axis of symmetry.

a) $y = x^2 - 3$

b) $y = (x + 1)^2 - 1$

c) $y = -2(x - 4)^2 - 3$

d) $y = 0.5(x + 6)^2 + 3$

37. The height of lava ejected from the Stromboli volcano can be modelled by the relation $h = -5t(t - 11)$, where h is the height, in metres, of the lava above the crater and t is the time, in seconds, since it was ejected.

a) Graph the relation.

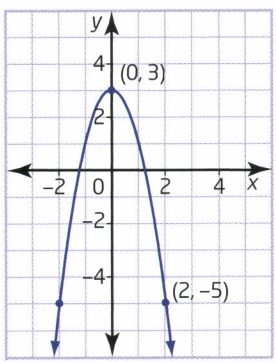
b) Find the maximum height reached by the lava, to the nearest metre.

c) How long does the lava take to reach the maximum height?

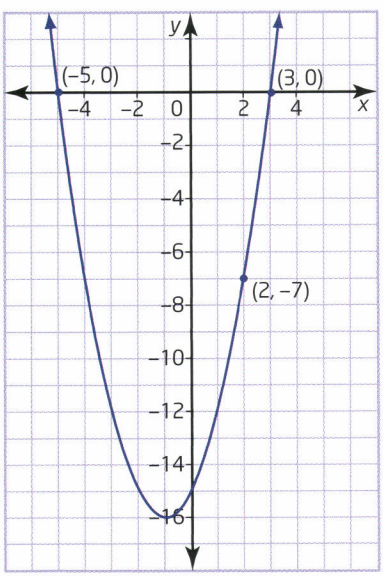
d) Is the length of time that the lava is in the air twice the answer from part c)? Explain why or why not.

38. Determine an equation to represent each parabola.

a)



b)



39. The vertex of a parabola is $(-2, -4)$. One x-intercept is 7. What is the other x-intercept?

40. Evaluate.

a) 8^0

b) 3^{-1}

c) $(-2)^{-5}$

d) $\left(\frac{2}{5}\right)^{-2}$

e) $(-18)^0$

f) $\left(\frac{4}{3}\right)^{-1}$

41. A piece of wood burns completely in 1 s at 600°C. The time it takes for the wood to burn is doubled for every 10°C drop in temperature and halved for every 10°C increase in temperature. Determine how long the piece of wood would take to burn at each temperature.

a) 500°C

b) 650°C

Chapter 5 Quadratic Expressions

42. Expand and simplify.

a) $3(x - 4) + 5(x + 6)$

b) $6(a + 3) - 2(a - 5)$

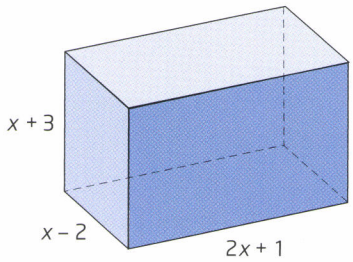
c) $4(1 - 3k) - (2 - 4k)$

d) $2t(3t - 4) + t(2t + 5)$

e) $\frac{1}{2}(6 + 3p) - \frac{3}{4}(8 - 10p)$

f) $3y(y^2 - y - 1) - y(2y^2 - 3y + 4)$

43. **a)** Write an algebraic expression to represent the surface area of the rectangular prism. Expand and simplify.



b) If x represents 5 cm, what is the surface area?

44. Expand and simplify.

a) $(x + 4)^2$

b) $(y - 4)(y + 4)$

c) $(a - 5)^2$

d) $(3t + 1)(3t - 1)$

e) $(5a + 3b)(5a - 3b)$

f) $2(3m + 1)^2$

45. Expand and simplify.

a) $(m - 3)(m + 3) + (m - 4)^2$

b) $3(2t + 1)^2 + 2(3t - 1)(3t + 1)$

c) $2(3x + 2y)(3x - 2y) - 3(3x - y)^2$

d) $(2y - 1)^2 + (y + 2)^2 - (1 - 2y)(1 + 2y)$

e) $(4m + n)^2 + 2(m - 2n)^2$

f) $5(2t - 5z)^2 + 3(4t - 3z)(4t + 3z)$

46. Factor.

a) $5k - 35$

b) $4h^2 - 20h$

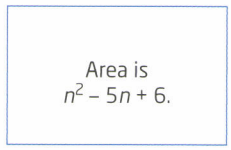
c) $2xy - 8xy^2$

d) $x^2 - 25$

e) $1 - 49m^2$

f) $4a^2 - 16b^2$

47. The area of the rectangle is given by the expression $n^2 - 5n + 6$.



a) Find expressions for the length and the width.

b) If n represents 8 cm, determine the perimeter and the area of the rectangle.

48. Factor each trinomial, if possible.

a) $x^2 - x - 12$

b) $y^2 + 3y - 18$

c) $m^2 + 11m + 24$

d) $t^2 - 8t + 15$

e) $x^2 + 3x + 4$

f) $n^2 - 13n + 40$

g) $w^2 - w - 30$

h) $14 + 5m - m^2$

49. Factor, if possible.

a) $x^2 + 10x + 25$

b) $y^2 - 12y + 36$

c) $m^2 + 6m + 16$

d) $4x^2 + 12x + 9$

e) $25r^2 - 20rs + 4s^2$

f) $5x^2 - 20xy + 20y^2$

50. Write two different trinomials that have $x + 2$ as a factor.

51. Determine all values of p so that each trinomial is a perfect square.

a) $x^2 + px + 16$

b) $4x^2 - 12x + p$

c) $px^2 + 40x + 16$

52. If m and n are integers, find values of m and n such that $m^2 - n^2 = 21$.

Chapter 6 Quadratic Equations

53. Rewrite each relation in the form $y = a(x - h)^2 + k$ by completing the square. Graph the relation and give the vertex and the equation of the axis of symmetry.

a) $y = x^2 + 4x + 1$

b) $y = -x^2 - 6x - 5$

c) $y = 3 - 4x - x^2$

54. Use Technology Use a graphing calculator to find the maximum or minimum point of each parabola, rounded to the nearest tenth.

a) $y = 2x^2 + 6x + 5$

b) $y = -3x^2 + 2x + 1$

c) $y = 0.8x^2 + 0.5x - 3.6$

55. Find the x-intercepts of each parabola.

a) $y = x^2 + 2x - 3$

b) $y = x^2 + 6x + 5$

c) $y = x^2 - 4x + 4$

d) $y = 4x^2 - 12x + 9$

56. Solve by factoring. Check your solutions.

a) $x^2 + 3x - 28 = 0$

b) $m^2 + 7m + 10 = 0$

c) $2n^2 = 27 - 15n$

d) $3k(k - 4) + k + 4(k + 1) = 0$

57. Write a quadratic equation with the given roots.

a) $5, -2$ **b)** $-\frac{3}{4}, -\frac{2}{3}$

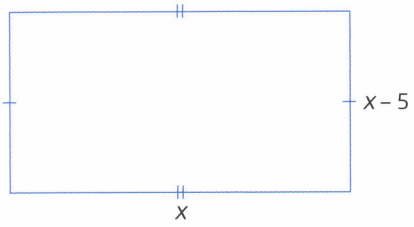
58. Find the value of k so that each parabola has only one x-intercept.

a) $kx^2 - 5x + 2 = 0$

b) $x^2 + kx + 9 = 0$

c) $25x^2 + 20x + k = 0$

59. The area of the rectangle is 36 cm^2. What are its dimensions?



60. Use the quadratic formula to solve each equation. Express your answers as exact roots.

a) $x^2 - x - 4 = 0$

b) $7k^2 - 2k - 2 = 0$

c) $2x^2 = 3 - 8x$

d) $4h = 5 - 4h^2$

e) $0 = -3a^2 + 4a + 1$

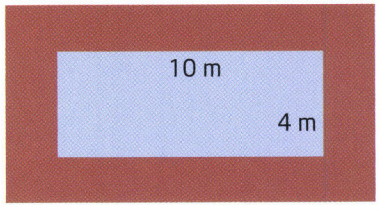
61. A store sells 90 ski jackets for \$200 each. A survey indicates that for each \$10 decrease in price, five more jackets will be sold.

a) Find the number of jackets sold and the selling price when the total revenue is \$17 600.

b) What is the lowest price that will give revenues of at least \$15 600? How many jackets would be sold at this price?

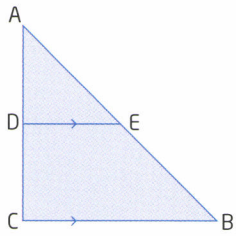
62. The hypotenuse of a right triangle measures 20 cm. The sum of the lengths of the legs is 28 cm. Find the length of each leg of the triangle.

63. A rectangular swimming pool measuring 10 m by 4 m is surrounded by a deck of uniform width. The total area of the pool and deck is 135 m^2. What is the width of the deck?



Chapter 7 Trigonometry of Right Triangles

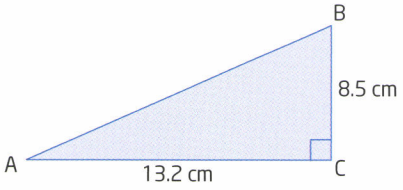
64. Identify the two similar triangles and explain why they are similar.



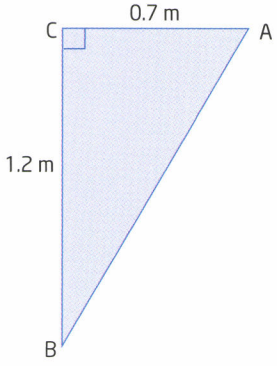
65. At noon, the shadow of a tree is 4.3 m long. At the same time, the shadow of a metre stick is 0.2 m long. What is the height of the tree?

66. Find the measure of $\angle A$, to the nearest degree.

a)



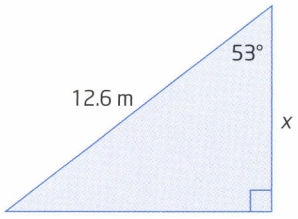
b)



67. A road sign shows that a hill has a grade of 8%. What is the angle of inclination of the hill, to the nearest tenth of a degree?

68. Find x, to the nearest tenth of a unit.

a)



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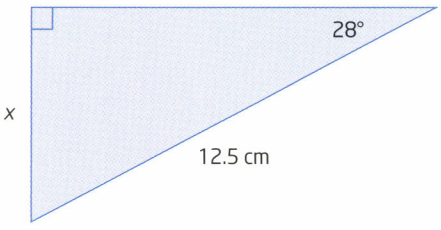
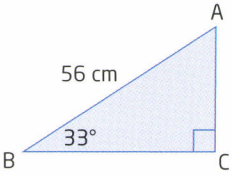


69. Solve each triangle. Round lengths to the nearest unit and angle measures to the nearest degree.

a)



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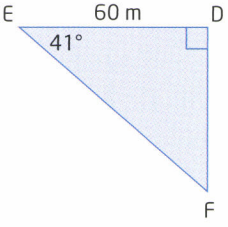

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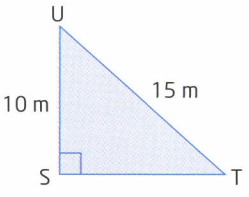

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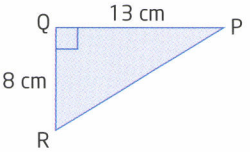


70. In $\triangle XYZ$, $\angle X = 90°$, XY = 3.5 cm, and YZ = 4.8 cm. Solve $\triangle XYZ$. Round lengths to the nearest tenth of a metre and angle measures to the nearest degree.

71. In $\triangle ABC$, $\angle C = 90°$. If $\sin A = 0.5$, what is the measure of $\angle B$?

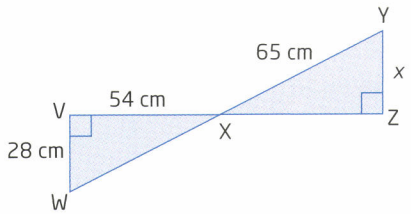
72. A coast guard boat is 14.8 km east of a lighthouse. A disabled yacht is 7.5 km south of the lighthouse.

a) How far is the coast guard boat from the yacht, to the nearest tenth of a kilometre?

b) At what angle south of due west, to the nearest tenth of a degree, should the coast guard boat travel to reach the yacht?

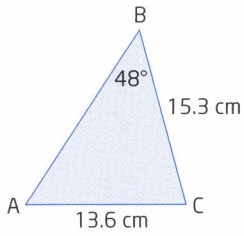
73. The Confederation Bridge joins New Brunswick and Prince Edward Island. From a boat in the Northumberland Strait, the angle of elevation of the highest point on the bridge is 26.6°. When the boat is 100 m closer to the bridge, the angle of elevation is 71.7°. What is the height of the bridge, to the nearest tenth of a metre?

74. Find the length of x, to the nearest centimetre.



Chapter 8 Trigonometry of Acute Triangles

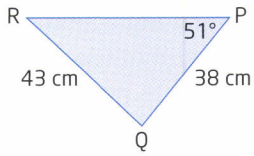
75. Find the measure of $\angle A$, to the nearest degree.



76. In $\triangle ABC$, $\angle B = 38.2°$, $\angle C = 65.6°$, and $b = 54$ cm. Find c, to the nearest tenth of a centimetre.

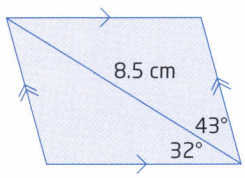
77. Solve each triangle. Round answers to the nearest tenth, if necessary.

a)

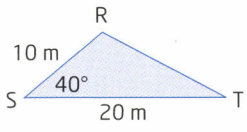


b) In $\triangle JKL$, $\angle L = 32°$, $j = 20.5$ cm, and $\angle K = 75°$.

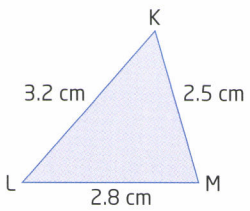
78. The longer diagonal of a parallelogram measures 8.5 cm. This diagonal makes angles of 43° and 32° with the sides of the parallelogram, as shown. Find the length of each side of the parallelogram, to the nearest tenth of a centimetre.



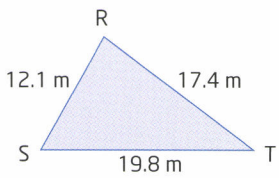
79. Find the unknown side length, to the nearest tenth of a metre.



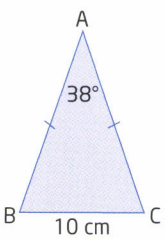
80. Solve $\triangle KLM$. Round answers to the nearest tenth of a degree.



81. Determine the area of $\triangle RST$, to the nearest square metre.

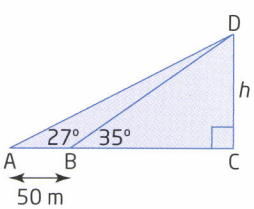


82. Determine the perimeter of isosceles $\triangle ABC$, to the nearest tenth of a centimetre.

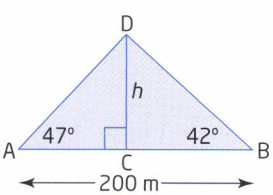


83. Find the length h in each, to the nearest tenth of a metre.

a)



b)



84. Two boats left the harbour at the same time. One travelled at 10 km/h on a bearing of N47°E. The other travelled at 8 km/h on a bearing of N79°E. How far apart were the boats after 45 min? Round the distance to the nearest tenth of a kilometre.

85. Solve the following triangles. Round lengths to the nearest tenth of a metre and angle measures to the nearest degree.

a) In $\triangle DEF$, $\angle D = 58°$, $\angle E = 53°$, and $d = 8$ cm.

b) In $\triangle RST$, $\angle R = 73°$, $r = 8$ m, and $t = 6$ m.

c) In $\triangle ABC$, $\angle A = 68°$, $b = 5$ cm, and $c = 7$ cm

d) In $\triangle WXY$, $w = 11$ m, $x = 10$ m, and $y = 14$ m

86. The three sides of a triangle measure 15 cm, 17 cm, and 18 cm. Find the measure of the largest angle, to the nearest degree.